J. Partial Diff. Eqs. **21**(2008), 59–76
 Vol.21 No.1

DISSIPATION AND DISPERSION APPROXIMATION TO HYDRODYNAMICAL EQUATIONS AND ASYMPTOTIC LIMIT*

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Dedicated to Professor Li Tatsien on the occasion of his seventieth birthday

(Received Oct. 16, 2007)

Abstract The compressible Euler equations with dissipation and/or dispersion correction are widely used in the area of applied sciences, for instance, plasma physics, charge transport in semiconductor devices, astrophysics, geophysics, etc. We consider the compressible Euler equation with density-dependent (degenerate) viscosities and capillarity, and investigate the global existence of weak solutions and asymptotic limit.

Key Words Hydrodynamics; degenerate viscosities; dispersion limit.
2000 MR Subject Classification 35B40, 82D37.
Chinese Library Classification O175.2.

1. Introduction

In the area of applied sciences, like plasma physics, transport of charged particles, astrophysics, geophysics, etc, the compressible Euler equations with additional dissipation of the form

$$\partial_t n + \nabla\cdot(nu) = 0, \tag{1.1}$$

$$\partial_t(nu) + \nabla\cdot(nu\otimes u) + \nabla p(n) = \rho F + f_{dis}, \tag{1.2}$$

are often used to simulate the dynamical behaviors of physical observable like the density $n > 0$, velocity u, momentum $J = nu$, and energy $e = e(n,u)$. Here, the Eq. (1.1) and (1.2) respectively express the conservation of mass and the balance of momentum. The force F is taken as the gradient filed of some potential $F = -\nabla\Phi$,

*L. Hsiao is supported by the NNSFC grant No.10431060. H.-L. Li is supported by the Beijing Nova Program 2005B48, the NNSFC grant No.10431060, the NCET support of the Ministry of Education of China, and the Re Shi Bu Ke Ji Ze You program.

where Φ represents either electrostatic potential or gravity, and can be determined by the self-consistent Poisson equation

$$\lambda\Delta\Phi = n \tag{1.3}$$

with $\lambda = \mp 1$. The term f_{dis} in (1.2) is chosen based on the different effects caused by the specific physical (dissipative or dispersive) mechanism, like drag friction (lubrication) $-n|u|u$ in the motion of shallow water [1], dispersion effects $\frac{\varepsilon^2}{2}\nabla(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}})$ with $\varepsilon > 0$ the scaled Planck constant in quantum mechanics [2], Korteweg term $\varepsilon n\nabla\Delta n$ with $\varepsilon > 0$ small parameter in phase transition [1], viscosity $\mu\Delta u$ or $\mu\nabla(n\nabla u)$ with μ viscosity coefficient in fluid-dynamics [3, 4], and so on.

The aim of this paper is to study the dissipative and dispersive approximation to the hydrodynamical system (1.1)–(1.2) as follows

$$\partial_t n + \mathrm{div}(nu) = 0, \tag{1.4}$$

$$\begin{aligned} &\partial_t(nu) + \mathrm{div}(nu\otimes u) + \nabla p(n) + n\nabla\Phi \\ &\quad = \varepsilon^2 n\nabla(\varphi'(n)\Delta\varphi(n)) + 2\eta\mathrm{div}\,(\mu(n)D(u)) + \eta\nabla(\lambda(n)\mathrm{div}u) - rn|u|u, \end{aligned} \tag{1.5}$$

where the right hand side terms in (1.5) consist of viscosity, dispersion and nonlinear friction, corresponding to the term f_{dis} in (1.2), and $D(u) = (\nabla u + {}^t\nabla u)/2$ is the stress tensor with degenerate viscosities $\mu(n) \geq 0$, $\lambda(n)$, and $\eta > 0$ a small parameter, which is zero in the appearance of vacuum $n = 0$. The nonlinear dispersion term is also taken into accounted with $\varphi(n) \geq 0$ and $\varepsilon > 0$ a small parameter, and the nonlinear term $-rn|u|u$ represents a drag friction with $r > 0$ a constant. The internal electrostatic potential Φ is chosen through the self-consistent Poisson equation

$$-\Delta\Phi = n - 1. \tag{1.6}$$

We consider the initial value problem of the approximate system (1.4)–(1.5) in $\mathbb{T}^N$ with initial data

$$n(x,0) = n_0(x), \quad nu(x,0) = m_0(x), \qquad x \in \mathbb{T}^N, \tag{1.7}$$

which satisfies

$$n_0 \geq 0 \;\text{ a.e. on }\; \mathbb{T}^N, \; \int n_0(x)\mathrm{d}x = 1, \;\text{and}\; \frac{|m_0|^2}{n_0} = 0 \text{ a.e. on } \{n_0(x) = 0\}. \tag{1.8}$$

The motivation to consider the approximate system (1.4)–(1.6) is the follows. Recently, the quantum hydrodynamic (QHD) model

$$\partial_t n + \nabla\cdot(nu) = 0, \tag{1.9}$$

$$\partial_t(nu) + \nabla\cdot(nu\otimes u) + \nabla p(n) = n\nabla\Phi + \frac{\varepsilon^2}{2}n\nabla(\frac{\triangle\sqrt{n}}{\sqrt{n}}), \tag{1.10}$$