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A REMARK ON THE BLOW-UP CRITERION OF STRONG SOLUTIONS AND REGULARITY FOR WEAK SOLUTIONS OF NAVIER-STOKES EQUATIONS*

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Abstract We give blow-up criteria of strong solutions of Navier-Stokes equations with its initial data in Besov spaces and consider the regularity of Leray-Hopf solutions of the equation.

Key Words Naiver-Stokes equations; blow-up; Littlewood-Paley decomposition; Leray-Hopf weak solution; Besov space.

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1. Introduction

In this paper, we consider Navier-Stokes equations (N-S) in $\mathbb{R}^n$, $n \geqslant 3$:

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0, \quad t>0, x\in\mathbb{R}^n \\ \mathrm{div} u = 0, \quad t>0, x\in\mathbb{R}^n \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $\nu > 0$, u and p denote the unknown velocity and pressure of incompressible fluid respectively.

It is given by Fujita-Kato [1] that for every $u_0(x) \in H^s \equiv W^{s,2}(\mathbb{R}^n)$ with $s > \frac{n}{2} - 1$, there exist $T = T(\|u_0\|_{H^s})$ and a solution $u(t)$ of N-S equation (1.1) on [0,T) in the class

$$(CL)_s \qquad u \in C([0,T);H^s)\cap C^1((0,T);H^s)\cap C((0,T);H^{s+2}).$$

There leaves a natural question whether the solution $u(t)$ loses its regularity at $t = T$. Giga [2] showed that if

$$(Se) \qquad \int_0^T \|u(t)\|_r^k \mathrm{d}t < \infty \quad \text{for} \quad \frac{2}{k}+\frac{n}{r} = 1 \quad \text{with} \quad n < r \leqslant \infty,$$

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then u can be continued to the solution in the class $(CL)_s$ beyond $t = T$. For the endpoint case $(r = \infty)$, Kozono and Taniuchi [3] proved if

$$\int_0^T \|u(t)\|_{BMO}^2 \mathrm{d}t < \infty,$$

then $u(t)$ can be continued to the strong solution in the class $CL_s(0, T')$ for some $T' > T$. Beale-Kato-Majda [4] proved that in $\mathbb{R}^3$, for Euler equations, if

$$\int_0^T \|\omega(t)\|_\infty \mathrm{d}t < \infty,$$

then $u(t)$ can never break down its regularity at $t = T$, here $\omega =$rot u. For the study of regularity of weak solutions, many mathematicians paid attention to the problem and obtained some partial results. We first recall the definitions of Leray-Hopf weak solution and strong solution and then state some known results. Here L^p_σ, H^1_σ respectively are the L^p and H^1 closures of $C^\infty_{0,\sigma} = \{f \in C^\infty_0, \mathrm{div} f = 0\}$ for $1 \leqslant p < \infty$.

Definition 1.1 *Let $u_0 \in L^2_\sigma$, a measurable function u on $\mathbb{R}^n \times (0,T)$ is called a weak solution of Navier-Stokes equations* (1.1) *if*
(i) $u \in L^\infty(0,T; L^2_\sigma) \cap L^2(0,T; H^1_\sigma)$;
(ii) $u(t)$ is continuous on [0,T] in the weak topology of L^2_σ ;
(iii) for every $0 \leqslant s \leqslant t < T$ and every $\Phi \in H^1((s,t); H^1_\sigma \cap L^n)$,

$$\int_s^t \{-(u, \partial_\tau \Phi) + (\nabla u, \nabla \Phi) + (u \cdot \nabla u, \Phi)\}\mathrm{d}\tau = -(u(t), \Phi(t)) + (u(s), \Phi(s)). \tag{1.2}$$

Definition 1.2 *Let $u_0 \in B^s_{p,q,\sigma}$(the same way to definite as L^p_σ) , for $s > \frac{n}{p} - 1$, a measurable function u on $\mathbb{R}^n \times (0,T)$ is called a strong solution of* (1.1) *if*
(i) $u \in C([0,T); B^s_{p,q,\sigma}) \cap C^1((0,T); B^s_{p,q,\sigma}) \cap C((0,T); B^{s+2}_{p,q,\sigma})$;
(ii) u satisfies (1.1) *with some distribution p such that $\nabla p \in C((0,T); B^s_{p,q})$.*

The weak solution for N-S equation is constructed under the energy space, first obtained by Leray [5] and Hopf [6] in the class $L^\infty(0,T; L^2(\mathbb{R}^n)) \cap L^2(0,T; \dot{H}^1(\mathbb{R}^n))$ and the solution solves the equation (1.1) in the sense of distribution. Early Prodi [7] Ohyama [8] and Serrin[9] obtained some partial regularity results for weak solution, then Giga[2] refined later, their results are: if Lerray's weak solution u satisfies

$$\int_0^T \|u(t)\|_r^k \mathrm{d}t < \infty \quad \text{for} \quad \frac{2}{k} + \frac{n}{r} = 1, \quad n < r \leqslant \infty,$$

then the solution is regular on $(0, T]$. Corresponding condition for the derivative of the solution is obtained by Beirão da Veiga [10, 11] through Sobolev embedding theorem, that is if

$$\int_0^T \left\| |\nabla|^s u(t) \right\|_r^k \mathrm{d}t < \infty \quad \text{for} \quad \frac{2}{k} + \frac{n}{r} = 1 + s, \quad \frac{n}{1+s} < r \leqslant \infty.$$