J. Partial Diff. Eqs. **21**(2008), 253–262

Vol.21 No.3

EXISTENCE AND NONEXISTENCE OF ENTIRE LARGE SOLUTIONS FOR SOME SEMILINEAR ELLIPTIC EQUATIONS*

Ye Dong
(Département de Mathématiques, UMR 8088, Université de Cergy-Pontoise,
95302 Cergy-Pontoise, France)
(E-mail: dong.ye@u-cergy.fr)
Zhou Feng
(Department of Mathematics, East China Normal University, Shanghai 200241, China)
(E-mail: fzhou@math.ecnu.edu.cn)

Dedicated to Professor Li Tatsien on the occasion of his 70th birthday

(Received Dec. 16, 2007)

Abstract In this note, we consider positive entire large solutions for semilinear elliptic equations $\Delta u = \rho(x)f(u)$ in $\mathbb{R}^N$ with $N \geq 3$. More precisely, we are interested in the link between the existence of entire large solution with the behavior of solution for $-\Delta u = \rho(x)$ in $\mathbb{R}^N$. Especially for the radial case, we try to give a survey of all possible situations under Keller-Osserman type conditions.

Key Words Entire large solution; existence criteria.
2000 MR Subject Classification 35J60, 35B05.
Chinese Library Classification O175.29, O175.25.

1. Introduction

We study the semilinear elliptic equation

$$\Delta u = \rho(x)f(u) \tag{1}$$

in the whole space $\mathbb{R}^N$ with $N \geq 3$, we are interested in the existence of positive solutions such that $\lim_{|x|\to\infty} u(x) = \infty$, the so called positive entire large solutions (*ELS* for shortness). Here ρ is a nontrivial nonnegative continuous function defined in $\mathbb{R}^N$. For fixing the idea, we assume always in this paper that

(H) $\quad f$ is continuous, nondecreasing and positive in $(0, \infty)$.

*This research is supported in part by NNSF No. 10671071 of China and the "basic research project of China".

When $\rho(x) \equiv 1$, thanks to the famous works of Keller [1] and Osserman [2], the existence of positive large solutions on regular bounded domains Ω is well understood now. That is, assuming moreover $f(0) = 0$ and f is locally lipschitz in $\mathbb{R}_+$, to get a positive solution of $\Delta u = f(u)$ in $\Omega \subset\subset \mathbb{R}^N$ with $\lim_{x\to\partial\Omega} u(x) = \infty$, the sufficient and necessary condition is

$$\mathcal{I}_{\sqrt{F}} := \int_1^\infty \frac{\mathrm{d}t}{\sqrt{F(t)}} < \infty \tag{2}$$

where

$$F(t) = \int_0^t f(s)\mathrm{d}s.$$

A lot of works has been done to consider the blow-up solutions for non autonomous situation $\Delta u = \rho(x)f(u)$ with a general bounded domain of $\mathbb{R}^N$. The existence, uniqueness or multiplicity, the asymptotic behavior or symmetry of large solution to (1) are somehow well understood now, when $\Omega \subset\subset \mathbb{R}^N$. Conversely, the general situation of *ELS* seems to be very open, even for the existence problem. Interested readers can find some recent developments in works listed at the end, and the references therein.

Many authors remarked that in the whole space situation, i.e. $\Omega = \mathbb{R}^N$, another quantity seems to play also an important role, which is

$$\mathcal{I}_f := \int_1^\infty \frac{\mathrm{d}t}{f(t)}. \tag{3}$$

The following lemma was given first in [3]. For the sake of completeness, we show here a short proof. In this note, C (or C_1, C_2) denotes a general positive constant, it could be changed from one line to another.

Lemma 1.1 *If f is positive in $(0, \infty)$ and nondecreasing near ∞, then $\mathcal{I}_{\sqrt{F}} < \infty$ implies that $\mathcal{I}_f < \infty$.*

Proof Since F is nondecreasing, it is well known that $\mathcal{I}_{\sqrt{F}} < \infty$ yields

$$\lim_{t\to\infty} \frac{t}{\sqrt{F(t)}} = 0.$$

So there exists some $t_0 > 0$ such that $t^2 \le F(t)$ for all $t \ge t_0$. As f is nondecreasing near ∞, for $t \ge t_0$ large enough, we get $F(t) \le 2tf(t)$,

$$F(t) \le 2\sqrt{F(t)}f(t), \quad \text{hence } \frac{1}{f} \le \frac{2}{\sqrt{F}} \text{ in } [t_0, \infty).$$

Therefore, we need to discuss three different cases: $\mathcal{I}_f = \infty$; $\mathcal{I}_{\sqrt{F}} = \infty$ but $\mathcal{I}_f < \infty$; or $\mathcal{I}_{\sqrt{F}} < \infty$.

As far as we know, no work has been done to give an exhaustive consideration for the three situations, even for the radial solution case. This is the main purpose of our study here.