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THE UNIQUENESS OF A TRANSONIC SHOCK IN A NOZZLE FOR THE 2-D COMPLETE EULER SYSTEM WITH THE VARIABLE END PRESSURE

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Dedicated to 70th birthday of Academian Li Daqian

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Abstract In this paper, by use of the methods in [1–3], we establish the uniqueness of a 2-D transonic shock solution in a nozzle when the end pressure in the diverging part of the nozzle lies in an appropriate scope. Especially, we remove the crucial but unnatural assumption in recent references which the transonic shock must be assumed to go through a fixed point in advance.

Key Words Steady Euler system; supersonic flow; subsonic flow; transonic shock; nozzle.

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1. Introduction and Main Results

In this paper, for the 2-D complete compressible Euler system, we are concerned with the uniqueness of a 2-D transonic shock solution in a nozzle when the variable end pressure in the diverging part of the nozzle lies in an appropriate scope. In [4–13] and [14–20], the authors have studied the well-posedness or ill-posedness of a transonic shock for the supersonic flow through a general 2-D or 3-D slowly-varying nozzle with an appropriately large exit pressure. However, the position of the shock in these references is either assumed to go through some fixed point in advance or determined by the ordinary differential equations which are resulted from the assumptions on symmetric properties of the supersonic coming flow, the nozzle walls and the end pressure. In this paper, under the more natural and physical boundary condition (that is, the position of the shock is really unknown other than it is assumed that the shock goes through some fixed point), we will study the uniqueness of a transonic shock with the appropriate end

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pressure. To do this, the essential difficulty comes from the complete uncertainty of the shock position. Thanks to the new ingredients used in [1–3], which only treats the isentropic Euler flow, we can show that the pressure along the nozzle wall is monotonic and the position of the transonic shock is uniquely determined by the end pressure. From this, the uniqueness of a transonic shock can be shown.

The two-dimensional complete Euler flow is governed by the following equations

$$\begin{cases} \partial_1(\rho u_1) + \partial_2(\rho u_2) = 0, \\ \partial_1(P + \rho u_1^2) + \partial_2(\rho u_1 u_2) = 0, \\ \partial_1(\rho u_1 u_2) + \partial_2(P + \rho u_2^2) = 0, \\ \partial_1((\rho e + \frac{1}{2}\rho|u|^2 + P)u_1) + \partial_2((\rho e + \frac{1}{2}\rho|u|^2 + P)u_2) = 0, \end{cases} \tag{1.1}$$

here $u = (u_1, u_2)$ is the velocity, ρ, P, e, S are the density, pressure, internal energy, specific entropy respectively. Moreover, the pressure function $P = P(\rho, S)$ and the internal energy function $e = e(\rho, S)$ are smooth in their arguments, in particular, $\partial_\rho P(\rho, S) > 0$ and $\partial_S e(\rho, S) > 0$ for $\rho > 0$. In this paper, for the convenience, we always write the state equations as $\rho = \rho(P, S)$ and $e = e(P, S)$. For the polytropic gas

$$P = A\rho^\gamma e^{\frac{S}{c_v}} \qquad \text{and} \qquad e = \frac{1}{\gamma - 1}\frac{P}{\rho},$$

here A, c_v and $\gamma > 1$ are positive constants, $c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)} = \sqrt{\frac{\gamma P}{\rho}}$ is called as the sonic speed.

We assume that the nozzle walls Γ_1 and Γ_2 are C^4-regular for $X_0 - 1 \le r = \sqrt{x_1^2 + x_2^2} \le X_0 + 1$ ($X_0 > 0$ is a fixed constant) and Γ_i consists of two curves Π_i^1 and Π_i^2, here Π_1^1 and Π_2^1 include the converging part of the nozzle, Π_1^2 and Π_2^2 construct a symmetric diverging part of the nozzle. More precisely, we assume that the equation of Π_i^2 is represented by $x_2 = (-1)^i \mathrm{tg}\theta_0 x_1$ with $x_1 > 0$ and $X_0 < r < X_0 + 1$, where $0 < \theta_0 < \frac{\pi}{2}$ is sufficiently small. Without loss of generality, we suppose that the C^3-smooth supersonic coming flow $(P_0^-(x), u_{1,0}^-(x), u_{2,0}^-(x), S_0^-)$ is symmetric near $r = X_0$, here $P_0^-(x) = P_0^-(r)$, $u_{i,0}^-(x) = \frac{U_0^-(r)x_i}{r}(i = 1, 2)$, and S_0^- is a constant (this assumption can be easily realized by the hyperbolicity of the supersonic coming flow and the symmetric property of the nozzle walls for $X_0 < r < X_0 + 1$, one can see [21]).

Suppose that the equation of the possible shock Σ is denoted by $x_1 = \psi(x_2)$, the flow field behind the shock Σ by $(P^+(x), u_1^+(x), u_2^+(x), S^+(x))$ and ahead of the shock front Σ by $(P^-(x), u_1^-(x), u_2^-(x), S^-(x))$. Then the Rankine-Hugoniot conditions on Σ