J. Partial Diff. Eqs. **21**(2008), 347–376

Vol.21 No.4

EXISTENCE AND UNIQUENESS RESULTS FOR VISCOUS, HEAT-CONDUCTING 3-D FLUID WITH VACUUM*

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Dedicated to Professor Li Daqian on the occasion of his 70th birthday

(Received Dec. 29, 2007)

Abstract We prove the local existence and uniqueness of the strong solution to the 3-D full Navier-Stokes equations whose the viscosity coefficients and the thermal conductivity coefficient depend on the density and the temperature. The initial density may vanish in an open set and the domain could be bounded or unbounded. Finally, we show the blow-up of the smooth solution to the compressible Navier-Stokes equations in $\mathbb{R}^n$ $(n \geq 1)$ when the initial density has compactly support and the initial total momentum is nonzero.

Key Words Compressible Navier-Stokes equations; existence; uniqueness; blow-up.

2000 MR Subject Classification 35Q30, 76N10.

Chinese Library Classification O175.

1. Introduction

The motion of a viscous, heat-conducting fluid in a domain $\Omega \subset \mathbb{R}^3$ can be described by the system of equations, known as the Navier-Stokes equations:

$$\rho_t + \mathrm{div}(\rho u) = 0, \tag{1.1}$$

$$(\rho u)_t + \mathrm{div}(\rho u \otimes u) - \mathrm{div}(\mu(\nabla u + \nabla u^\top)) - \nabla(\lambda \mathrm{div} u) + \nabla p = \rho f, \tag{1.2}$$

$$(\rho e)_t + \mathrm{div}(\rho e u) - \mathrm{div}(\kappa \nabla \theta) + p \mathrm{div} u = \frac{\mu}{2}|\nabla u + \nabla u^\top|^2 + \lambda(\mathrm{div} u)^2 + \rho h, \tag{1.3}$$

in $(0, +\infty) \times \Omega \subset (0, +\infty) \times \mathbb{R}^3$, and the initial and boundary conditions:

$$(\rho, \theta, u)|_{t=0} = (\rho_0, \theta_0, u_0) \text{ in } \Omega, \tag{1.4}$$

$$(\theta, u) = (\theta_b, 0) \text{ on } (0, \infty) \times \partial\Omega, \tag{1.5}$$

*This work is supported by NSFC 10571158, Zhejiang Association for international exchange of personal and China Postdoctoral Science Foundation 20060400335.

$$(\rho, \theta, u)(t, x) \to (\rho^\infty, \theta^\infty, 0) \text{ as } |x| \to \infty. \tag{1.6}$$

In this paper, Ω is either a bounded domain in $\mathbb{R}^3$ with smooth boundary or an usual unbounded domain such as the whole space $\mathbb{R}^3$, the half space $\mathbb{R}^3_+$ ($\theta_b = \theta^\infty$) and an exterior domain in $\mathbb{R}^3$ with smooth boundary. The known fields h and f denote a heat source and an external force per unit mass. ρ, θ, e, p and u denote the unknown density, temperature, internal energy, pressure and velocity fields of the fluid, respectively. The pressure p and internal energy e are related to the density and temperature via the equations of state:

$$p = p(\rho, \theta), \ \ e = e(\rho, \theta).$$

In accordance with the basic principles of classical thermodynamics, p and e are inter-related through the following relationship:

$$\partial_\rho e = \frac{1}{\rho^2}(p - \theta \partial_\theta p).$$

The viscosity coefficient $\mu = \mu(\rho, \theta)$, the second viscosity coefficient $\lambda = \lambda(\rho, \theta)$ and the thermal conductivity coefficient $\kappa = \kappa(\rho, \theta)$ are required to satisfy

$$\mu > 0, \ \ 3\lambda + 2\mu \geq 0 \quad \text{and} \ \ \kappa \geq 0.$$

In this paper, we assume that $(e, p, \kappa, \mu, \lambda)$ satisfy

$$\partial_\theta e(\rho, \theta) \geq \underline{c} > 0, \ \ 0 \leq p(\rho, \theta) \leq \rho^{\frac{1}{2}} p_0(\rho)(1 + \theta^2), \tag{1.7}$$

$$|\partial_\theta p(\rho, \theta)| \leq \rho^{\frac{1}{2}} p_1(\rho)(1 + \theta), \ \ |\partial_\rho p(\rho, \theta)| \leq p_2(\rho)(1 + \theta^2), \tag{1.8}$$

$$\mu(\rho, \theta) \geq \mu^0 > 0, \ \ 3\lambda(\rho, \theta) + 2\mu(\rho, \theta) \geq 0, \ \ \kappa(\rho, \theta) \geq \kappa^0 > 0, \tag{1.9}$$

$$\rho^{-\frac{1}{4}} \partial_\theta(\partial_\theta e, \partial_\rho e, \mu, \lambda, \kappa)(\rho, \theta) \in L^\infty_{loc}([0, \infty) \times [0, \infty)), \tag{1.10}$$

$$\partial_\theta e, \partial_\rho e, p, \mu, \lambda, \kappa \in C^1([0, \infty) \times [0, \infty)), \ \ p_0, p_1, p_2 \in C([0, \infty)), \tag{1.11}$$

and

$$\partial_\theta p, \partial_\rho p \in C^\alpha([0, \infty), L^\infty_{loc}([0, \infty))), \tag{1.12}$$

for all $\rho, \theta \geq 0$ and a constant $\alpha \in (\frac{1}{2}, 1)$. It is easy to see that the ideal flow ($p = R\rho\theta, e = C_v\theta$, and $C_v, \kappa, \mu, \lambda =$ constants) satisfies the above assumptions with $p_0 = p_1 = \rho^{\frac{1}{2}}$ and $p_2 = 1$. The above assumptions are motivated by the facts in [1] where it is pointed out that μ, λ and κ of a real gas are vary with the temperature and density, e grows as $\theta^{1+\gamma}$ with $\gamma \approx 0.5$ and κ increases like θ^q with $q \in [4.5, 5.5]$.

In the case that the data $(\rho_0, \theta_0, u_0, f, h)$ are sufficiently regular and the initial density ρ_0 is bounded away from zero, there exits a unique local strong solution to the problem (1.1)-(1.6), and the solution exists globally in time provided that the initial data are small in some sense. For details, we refer the readers to papers [2, 3] and the references therein. When μ, λ, κ are three constants, there have been some existence results on the strong solutions for the general case of nonnegative initial density. For details, we refer the readers to papers [4, 5] and the references therein.