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A Nonlinear Diffusion System with Coupled Nonlinear Boundary Flux

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Abstract. This paper studies a nonlinear diffusion system with coupled nonlinear boundary flux and two kinds of inner sources (positive for the first and negative for the second), where the four nonlinear mechanisms are described by eight nonlinear parameters. The critical exponent of the system is determined by a complete classification of the eight nonlinear parameters, which is represented via the *characteristic algebraic system* introduced to the problem.

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1 Introduction

In this paper we consider coupled nonlinear diffusion equations of the form

$$(u^m)_t=\Delta u+a_1u^{\alpha_1},\quad (v^n)_t=\Delta v-a_2v^{\beta_1},\qquad (x,t)\in\Omega\times(0,T),\tag{1.1}$$

$$\frac{\partial u}{\partial\eta}=u^{\alpha_2}v^p,\quad \frac{\partial v}{\partial\eta}=u^qv^{\beta_2},\qquad (x,t)\in\partial\Omega\times(0,T),\tag{1.2}$$

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\qquad x\in\bar{\Omega},\tag{1.3}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, the parameters m,n,p,q,a_1,a_2 are positive, $\alpha_i,\beta_i\geq 0$ $(i=1,2)$, u_0 and v_0 are positive functions satisfying

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the compatibility conditions on $\bar{\Omega}$. The diffusion in (1.1) may be fast or slow, e.g., when $m>1$ or $0<m<1$ for the component u. There are positive and negative sources for u and v respectively, together with coupled nonlinear boundary condition (1.2) describing the nonlinear radiation laws in heat propagations. The critical exponent to the semilinear case $(m=n=1)$ of (1.1)-(1.3) was studied by Zheng, Liang and Song [1]. They introduced the matrix equation

$$\begin{pmatrix}\alpha_2-\mu & p\\ q & \beta_2-\gamma\end{pmatrix}\begin{pmatrix}\rho_1\\ \rho_2\end{pmatrix}=\begin{pmatrix}1\\ 1\end{pmatrix}$$

with

$$\mu=1-\Big(\frac{\alpha_1-1}{2}\Big)_+,\quad \gamma=1+\Big(\frac{\beta_1-1}{2}\Big)_+,$$

and obtained that the critical exponent of (1.1)-(1.3) is just $(1/\rho_1,1/\rho_2)=(0,0)$. We refer also to, e.g., Zheng *et al*. [2, 3], Bedjaouit and Souplet [4] for the results on parabolic equations with inner absorptions.

The scalar case was studied by Filo [5], and Deng *et al*. [6]. For the scalar nonlinear diffusion equation with positive source

$$(u^m)_t=\Delta u+a_1u^\alpha,\qquad (x,t)\in\Omega\times(0,T),\tag{1.4}$$

$$\frac{\partial u}{\partial\eta}=b_1u^\beta,\qquad (x,t)\in\partial\Omega\times(0,T),\tag{1.5}$$

$$u(x,0)=u_0(x),\qquad x\in\bar{\Omega},\tag{1.6}$$

with $m>0$, $\alpha,\beta\geq 0$, $a_1,b_1>0$, Song and Zheng [7] proved the following result:

Proposition 1.1. *The solutions of* (1.4)-(1.6) *blow up in a finite time for large initial value provided* (i) $\alpha>m$, $a_1>0$, *or* (ii) $0<m\leq 1$, $\beta>m$, $b_1>0$, *or* (iii) $m>1$, $\beta>(m+1)/2$, $b_1>0$.

Andreu *et al*. [8] and Li *et al*. [9] studied the scalar case with absorption, i.e.,

$$(u^n)_t=\Delta u-a_2u^{\beta_1},\qquad (x,t)\in\Omega\times(0,T),\tag{1.7}$$

$$\frac{\partial u}{\partial\eta}=u^{\beta_2},\qquad (x,t)\in\partial\Omega\times(0,T),\tag{1.8}$$

$$u(x,0)=u_0(x),\qquad x\in\bar{\Omega},\tag{1.9}$$

with $a_2,n>0$, $\beta_i\geq 0$ $(i=1,2)$, and obtained the blow-up criterion:

Proposition 1.2. *The solutions of* (1.7)-(1.9) *blow up in a finite time for large initial data if*

(i) $n\geq 1$ *with* $\beta_1\leq n$, $\beta_2>(n+1)/2$ *or* $\beta_1>n$, $\beta_2>(\beta_1+1)/2$; *or*

(ii) $0<n<1$ *with* $\beta_2>1$, $\beta_2>(\beta_1+1)/2$, *or* $n<\beta_2\leq 1$, $\beta_2>\beta_1$.

Recently, Zheng and Wang [10] established the critical exponent for the nonlinear diffusion system with inner absorptions and nonlinear boundary conditions.