JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
Vol. **22**, No. 1, pp. 1-10

J. Part. Diff. Eq.
February 2009

# Dirichlet Eigenvalue Ratios for the $p$-sub-Laplacian in the Carnot Group

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Received 16 November, 2007; Accepted 29 August, 2008

**Abstract.** We prove some new Hardy type inequalities on the bounded domain with smooth boundary in the Carnot group. Several estimates of the first and second Dirichlet eigenvalues for the $p$-sub-Laplacian are established.

**AMS Subject Classifications**: 35B05, 35H99

**Chinese Library Classifications**: O175.24

**Key Words**: Carnot group; $p$-sub-Laplacian; Dirichlet eigenvalue; Hardy-type inequality.

## 1 Introduction

Various boundary value problems on bounded domains in the Euclidean space for the Laplacian and $p$-Laplacian and their applications in nonlinear problems have been studied extensively, see [1,2] and references therein. Boundary value problems (including the Dirichlet eigenvalue problem) for the sub-Laplacian in the Heisenberg group and Carnot groups have also received some attention in recent years, see, e.g., [3, 4] and references therein. However, we have not seen the results for the Dirichlet eigenvalue problem of the $p$-sub-Laplacian ($p>1$) in the Carnot group.

In this paper, we consider the ratio of the first and second eigenvalues for the Dirichlet problem,

$$\begin{cases} -\Delta_{G,p}u=\lambda|u|^{p-2}u, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega$ is a bounded domain with smooth boundary in the Carnot group $G$, $\Delta_{G,p}$ is the $p$-sub-Laplacian in $G$ with the form

$$\Delta_{G,p}u=\nabla_G\cdot(|\nabla_G u|^{p-2}\nabla_G u),\ \ p>1.$$

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Here $\nabla_G u=(X_1u,\cdots,X_mu)$, $\{X_j\}_{j=1}^m$is a left-invariant basis of the first floor of the Lie algebra corresponding to the Carnot group.

**Definition 1.1.** *A pair* $(u,\lambda)\in W_0^{1,p}(\Omega)\times\mathbb{R}$ *is a weak solution of the Dirichlet problem* (1.1) *provided that*

$$\int_\Omega|\nabla_G u|^{p-2}\langle\nabla_G u,\nabla_G v\rangle\mathrm{d}x=\lambda\int_\Omega|u|^{p-2}uv\mathrm{d}x, \tag{1.2}$$

*for any* $v\in W_0^{1,p}(\Omega)$*, where such a pair*$(u,\lambda)$*, with u nontrivial, is called an eigenpair;* $\lambda$ *is an eigenvalue and u is called an associated eigenfunction. By choosing* $v=u$ *in* (1.2)*, it follows that all eigenvalues* $\lambda$ *are nonnegative.*

The arguments as in the Euclidean space show easily that the existence of eigenvalues, simplicity of the first eigenvalue in (1.1) and the variational characterization of the second eigenvalue are true. The authors in [2] provided the fundamental eigenvalue ratio of the $p$-Laplacian in the Euclidean space. We hope to give such estimates for the $p$-sub-Laplacian in the Carnot group $G$. In Section 2, some relevant facts on the Carnot group are presented. Nevertheless, when the method in [2] is used to our case, some new difficulties appear. For our purpose, in Section 3, several Hardy-type inequalities are established. Note that D'Ambrosio [5] obtained the following inequality on bounded domains in $G$

$$c\int_\Omega\frac{|u|^p}{\phi^p}|\nabla_G\phi|^p\mathrm{d}x\le\int_\Omega|\nabla_G u|^p\mathrm{d}x,$$

where $u\in W_0^{1,p}(\Omega)$, $\phi$ is some weight function such that

$$-\Delta_{G,p}\phi=\nabla_G(|\nabla_G\phi|^{p-2}\cdot\nabla_G\phi)\ge0.$$

Because of the appearance of the weight $|\nabla_G\phi|^p$, we find that such class of inequalities cannot be applied to estimate eigenvalues in our case. We also relate a useful property of the Sobolev space $W_0^{1,p}(\Omega)$ in this section. The final section is devoted to the estimates for the first and second eigenvalues based on the results above.

## 2 Preliminaries

We collect some notations and properties for the Carnot group (see, e.g., [6–8]).

The Carnot group $G=(\mathbb{R}^n,\cdot)$ is a connected and simply connected nilpotent Lie group whose Lie algebra $\mathfrak{g}$ possesses a stratification, i.e., there exist linear subspaces $V_1,\cdots,V_k$ of $\mathfrak{g}$ such that

$$\mathfrak{g}=V_1\oplus\cdots\cdots\oplus V_k,\quad [V_1,V_i]=V_{i+1},\quad i=1,\cdots,k-1,\ \text{ and }\ [V_1,V_k]=0,$$

where $[V_1,V_i]$ is the subspace of $\mathfrak{g}$ generated by the elements $[X,Y]$ with $X\in V_1,Y\in V_i$. In this way we get a Carnot group of step $k$ and the integer $k\ge1$ is the step of $G$.