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Traveling Wave Solutions to the Three-Dimensional Nonlinear Viscoelastic System

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Abstract. In this paper, we establish the existence of traveling wave solutions to the nonlinear three-dimensional viscoelastic system exhibiting long range memory. Under certain hypotheses, if the speed of propagation is between the speeds determined by the equilibrium and instantaneous elastic tensors, then the system has nontrivial traveling wave solutions. Moreover, the system has only trivial traveling wave solution in some cases.

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1 Introduction

In this paper, we discuss the existence of nontrivial traveling wave solutions to the three-dimensional nonlinear viscoelastic system exhibiting long range memory:

$$\boldsymbol{u}_{tt}(\boldsymbol{x},t)=\mathrm{div}_x\boldsymbol{\sigma}, \tag{1.1a}$$

where $\boldsymbol{u}(\boldsymbol{x},t)=(u_1(\boldsymbol{x},t),u_2(\boldsymbol{x},t),u_3(\boldsymbol{x},t))$ is the displacement of a material particle $\boldsymbol{x}=(x_1,x_2,x_3)$ at time t, and $\boldsymbol{\sigma}=(\sigma_{ij})$ is the stress tensor. For viscoelastic materials, the stress at time t depends on all the history of the deformation gradient up to time t. Here we discuss only the case when the stress is given by a single integral law (see, [1,2])

$$\boldsymbol{\sigma}(\boldsymbol{x},t)=\boldsymbol{g}(\nabla\boldsymbol{u}(\boldsymbol{x},t))-\int_0^\infty \boldsymbol{h}(\tau,\nabla\boldsymbol{u}(\boldsymbol{x},t-\tau))\mathrm{d}\tau, \tag{1.1b}$$

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where $\boldsymbol{g}(\boldsymbol{\xi})=(g_{ij}(\boldsymbol{\xi})),\boldsymbol{h}(\tau,\boldsymbol{\eta})=(h_{ij}(\tau,\boldsymbol{\eta})),\boldsymbol{\xi}=(\xi_{ij}),\boldsymbol{\eta}=(\eta_{ij}),\ i,j=1,\ 2,\ 3.$

Our interest is to find traveling wave solutions to the nonlinear Volterra integro-differential system (1.1). That is, we are looking for a solution $\boldsymbol{u}(\boldsymbol{x},t)$ depending only on $\xi=\lambda t+\boldsymbol{\omega}\cdot\boldsymbol{x}$, where λ is the speed of propagation, $\boldsymbol{\omega}=(\omega_1,\omega_2,\omega_3)$ with $|\boldsymbol{\omega}|=1$. Moreover, we require that the solution satisfies the upstream condition:

$$\lim_{\xi\to-\infty}\frac{\mathrm{d}\boldsymbol{u}(\xi)}{\mathrm{d}\xi}=\boldsymbol{v}^- \tag{1.2}$$

for a given constant vector $\boldsymbol{v}^-=(v_1^-,v_2^-,v_3^-)$.

Qin and Ni studied the special case in [3] when

$$\boldsymbol{h}(\tau,\nabla\boldsymbol{u}(\boldsymbol{x},t-\tau))=a(\tau)\boldsymbol{h}(\nabla\boldsymbol{u}(\boldsymbol{x},t-\tau)). \tag{1.3}$$

As pointed out in [3], for the pure elastic case

$$\boldsymbol{\sigma}=\boldsymbol{g}(\nabla\boldsymbol{u}(\boldsymbol{x},t)), \tag{1.4}$$

the problem has no nontrivial traveling wave solution except when λ is the speed of propagation for the wave $\boldsymbol{v}^-$ determined by the elastic tensor. For viscoelastic materials, the instantaneous elastic tensor (1.4) is different from the equilibrium elastic tensor determined by the stress tensor

$$\boldsymbol{p}(\boldsymbol{u}(\boldsymbol{x},t))=\boldsymbol{g}(\nabla\boldsymbol{u}(\boldsymbol{x},t))-\int_0^\infty\boldsymbol{h}(\tau,\nabla\boldsymbol{u}(\boldsymbol{x},t))\mathrm{d}\tau, \tag{1.5}$$

which governs the long time behavior of waves. Thanks to the dissipative effect, in general, the speed of propagation for the wave determined by the equilibrium elastic tensor is less than that determined by the instantaneous elastic tensor. Therefore, we should find nontrivial traveling wave solutions to the problem with the propagation speed λ between the two speeds.

For the one-dimensional case, the system (1.1) is reduced to

$$u_{tt}(x,t)=\frac{\partial}{\partial x}g(u_x(x,t))-\int_0^\infty\frac{\partial}{\partial x}h(\tau,u_x(x,t-\tau))\mathrm{d}\tau, \tag{1.6}$$

and the corresponding instantaneous and equilibrium elastic modulus are $g'(u_x)$ and $p'(u_x)$, respectively. The authors of [4] and [5] proved that if

$$p'(v^-)<\lambda^2<g'(v^-), \tag{1.7}$$

then there exist nontrivial traveling wave solutions to (1.6).

All the methods used in the one-dimensional case depend strongly on the monotonicity of both traveling wave solutions and iterative sequences. Therefore, they cannot be applied to the three-dimensional case. In order to overcome this difficulty, we apply the