JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
Vol. **22**, No. 2, pp. 97-110

J. Partial Diff. Eq.
May 2009

Modified Boussinesq System with Variable Coefficients: Classical Lie Approach and Exact Solutions

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Received 31 January 2007; Accepted 6 November 2008

Abstract. The Lie-group formalism is applied to investigate the symmetries of the modified Boussinesq system with variable coefficients. We derived the infinitesimals and the admissible forms of the coefficients that admit the classical symmetry group. The reduced systems of ordinary differential equations deduced from the optimal system of subalgebras are further studied and some exact solutions are obtained.

AMS Subject Classifications: 35Q53, 37J15, 35G99

Chinese Library Classifications: O175.29

Key Words: Lie symmetries; nonlinear diffusion; exact solutions; optimal system; reductions; global solutions; characteristic algebraic system.

1 Introduction

The nonlinear evolution equations have numerous applications in physical sciences and engineering. The Boussinesq equation is a well known evolution equation governing many physical systems. It was first introduced by Boussinesq in 1871 to describe the propagation of long waves in shallow water [1]. It also arises in several other physical applications including one-dimensional nonlinear lattice waves [2], vibrations in a nonlinear string [3], and iron sound waves in a plasma [4]. Among various forms of the Boussinesq equations, one is modified Boussinesq system that was first introduced by Foddy and Gibbons [5] and then studied by other researchers [6–8]. However, the

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physical situations in which nonlinear equations arise tend to be highly idealized due to assumption of constant coefficients.Due to this, much attention has been paid to the study of nonlinear equations with variable coefficients [9–17].

Often, it is very difficult to solve explicitly these nonlinear equations for exact solutions. Consequently perturbation, asymptotic and numerical methods are applied to obtain approximate solutions of these equations. However, there is much current interest in obtaining exact solutions of these equations; these solutions provide much information about nonlinear phenomena and well described various aspects of the physical phenomena; these solutions are useful to discuss and examine the sensitivity of physical phenomena with several important parameters described by variable coefficients. The exact solutions are also helpful in designing and testing of numerical algorithm. Mathematical techniques which generate a wide range of solutions and applicable to all type of nonlinear differential equations are few. The group theoretic techniques can be categorized in this class and generally it produces a variety of exact solutions. Lie group method [18–21] is an effective and simplest method among group theoretic techniques and a large number of equations are solved with the aid of this method, some recent contributions can be seen in the articles [22–28]. In this paper, we have studied the variable coefficients version of the modified Boussinesq system

$$u_t+\alpha_1(t)uv_x+\alpha_2(t)u_xv+\alpha_3(t)v_{xx}=0, \tag{1.1a}$$

$$v_t+\beta_1(t)uu_x+\beta_2(t)vv_x+\beta_3(t)u_{xx}=0, \tag{1.1b}$$

for exact solutions with the help of Lie's classical method [18].

The paper has been organized as follows. Section 2 is devoted to the outline of Lie group method to generate various symmetries of the Boussinesq system, and an optimal system comprising five basic vector fields is identified. Section 3 contains the reduced systems of ordinary differential equations (ODEs) and their exact solutions. Final section is for discussion.

2 Lie symmetry group

Lie's method of infinitesimal transformation groups which essentially reduces the number of independent variables in partial differential equation (PDE) and reduces the order of ODE, has been widely used in the equations of mathematical physics. The classical method for finding symmetry reductions of PDEs is the Lie group method of infinitesimal transformations and the associated determining equations are an over determined linear system. The technique has earlier been used to obtain the exact solutions of various nonlinear partial differential equations, hence there is no need to discuss this method in detail. In this section, we will obtain the symmetry groups using the Lie's classical method. The method mainly consists of the following steps: