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The Regularity of a Class of Degenerate Elliptic Monge-Ampère Equations

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Abstract. In the present paper the regularity of solutions to Dirichlet problem of degenerate elliptic Monge-Ampère equations is studied. Let $\Omega\subset\mathbb{R}^2$ be smooth and convex. Suppose that $u\in C^2(\overline{\Omega})$ is a solution to the following problem: $\det(u_{ij})=K(x)f(x,u,Du)$ in Ω with $u=0$ on $\partial\Omega$. Then $u\in C^\infty(\overline{\Omega})$ provided that $f(x,u,p)$ is smooth and positive in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2$, $K>0$ in Ω and near $\partial\Omega$, $K=d^m\tilde{K}$, where d is the distance to $\partial\Omega$, m some integer bigger than 1 and $\tilde{K}$ smooth and positive on $\overline{\Omega}$.

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1 Introduction

The study of the Monge-Ampère equations

$$\det(\mathrm{D}^2u)=\det(u_{ij})=K(x)f(x,u,\mathrm{D}u) \quad \text{in } \Omega \tag{1.1}$$

arises from many problems in differential geometry, mass transportation, fluid dynamics and so on. Throughout the present paper we always assume $f(x,z,p)$ strictly positive and smooth in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$. If $K(x)\in C^\infty(\overline{\Omega})$ is strictly positive, (1.1) is fully nonlinear elliptic and the regular theorem on fully nonlinear elliptic equations tells us that any convex solution in $C^2(\Omega)$ to (1.1) is indeed in $C^\infty(\Omega)$. As far as the boundary value problem is concerned, the existence of smooth convex solutions is obtained for both of the Dirichlet problem and the Neumann problem in [1–3] and so on. These results show that any convex solution in $C^2(\overline{\Omega})$ is in $C^\infty(\overline{\Omega})$ too, provided $K\geq c_0>0,f,\partial\Omega$ and the boundary data are all smooth.

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However, if $K(x)$ is only assumed to be non-negative, (1.1) is degenerate elliptic and the situation is quite complicated. A well-known example that $u=|x|^{2+2/n}$ solves (1.1) with $K=|x|^2$ and f being some constant, tells us that even the right hand side of (1.1) is analytic, we still cannot expect the solutions to have good regularity. In [4], there is also a solution in $C^{2,1}\backslash C^3$ of (1.1) defined on S^2 with analytic f and K subject to f strictly positive and K non-negative, vanishing only at some point $p\in S^2$. Generally speaking, we cannot expect smooth solutions to degenerate elliptic equations without special additional assumptions. In this direction, when $K\equiv 0$, [5] gives the existence of $C^{0,1}(\overline{\Omega})$ solutions for Dirichlet problems. [6] proves that the homogeneous Dirichlet problem of (1.1) possess a solution in $C^{1,1}(\overline{\Omega})$ if $f\equiv 1$ and $K\geq 0$, $K^{1/(n-1)}\in C^{1,1}(\overline{\Omega})$. The counter example in [7] indicates that this result is sharp if no other assumptions are imposed. However, for the practical applications, we are often facing to solve degenerate elliptic Monge-Ampère equations in spaces of functions of higher regularity than $C^{1,1}$, and moreover, sometimes in C^∞. Therefore it is worth studying the following problem,

To find some sufficient condition so that any C^2 (even $C^{1,1}$) solutions of (1.1) *are smooth.*

In two dimensions, by means of the Ampère transformation (also called partial Legendre transformation, see [8]) and estimates of subelliptic operators, Guan in [9], and Guan and Sawyer in [10] have given an affirmative answer that any solution in $C^{1,1}(\Omega)$ is in $C^\infty(\Omega)$ if $\Delta u>0$ in Ω and K is smooth and degenerate at finite degree. In [11,12], Rios, Sawyer and Wheeden have generalized the Ampère transformation to higher dimensions n and obtained the smoothness of C^2 solutions if the rank of Hessian D^2u equals to $(n-1)$ at every points where the equation is degenerate. The present paper intends to deal with the case where degeneracy only occurs on the boundary and studies the smoothness of C^2 solutions. Such cases are often encountered in the applications. For example, to find a smooth isometric embedding of a given smooth nonnegatively curved metric, see, e.g., [13–15], one must seek a smooth solution to (1.1) with boundary condition

$$u=0 \quad \text{on } \partial\Omega. \tag{1.2}$$

In [16], Hong and Zuily proved that in two dimensions, any solution in $C^2(\overline{\Omega})$ to (1.1) with (1.2) is in $C^\infty(\overline{\Omega})$ if f is strictly positive and smooth in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^n$, $\partial\Omega$ smooth and strictly convex, $K>0$ in Ω and $K=0\neq|dK|$ on $\partial\Omega$. Moreover, combining with the standard C^2 estimates on the boundary Hong [17] presented the existence of solutions smooth up to the boundary provided that f satisfies some natural structure conditions. The purpose of the present paper is to generalize the results in [16] to the cases where K degenerates at finite degree on the boundary. More precisely, we will prove

Theorem 1.1. *Suppose that $f(x,z,p)$ is positive and smooth in $\overline{\Omega}\times\mathbb{R}\times\mathbb{R}^2$, with Ω smooth and strictly convex, and that*

$$K=|\phi|^m\bar{K} \tag{1.3}$$

on $\overline{\Omega}$ for some integer $m\geq 2$, where ϕ is the defining function of $\partial\Omega$ and $\bar{K}$ is positive and smooth on $\overline{\Omega}$. Then any solution in $C^2(\overline{\Omega})$ to (1.1) *with* (1.2) *is in fact in $C^\infty(\overline{\Omega})$.*