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An Image Denoising-Deblurring Adaptive Variational Problem in BV Space

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Abstract. In this paper, we establish the existence and uniqueness of a BV solution to an initial boundary value problem for a nonlinear integro-differential equation, which is related to a denoising and deblurring variational model in image restoration. Several experiments relevant to the restoration model are performed and numerical results are discussed.

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1 Introduction

The image restoration is to recover a "true" image u from an observed image u_0 defined on a bounded domain $\Omega\subset\mathbb{R}^N$ with piecewise Lipschitz boundary, and the latter is usually modeled by

$$u_0=\mathcal{K}u+\eta, \tag{1.1}$$

where $\mathcal{K}$ is the blur operator, η is the additive noise. In order to approximate u from the observed image u_0, the optimization is reduced to the following minimizing problem [1],

$$\min\int_\Omega[\psi(x,|\nabla u|)+\frac{\lambda}{2}|\mathcal{K}u-u_0|^2]\mathrm{d}x. \tag{1.2}$$

The corresponding evolution problem for (1.2) is

$$u_t-\mathrm{div}\Big(\psi_r'(x,|\nabla u|)\frac{\nabla u}{|\nabla u|}\Big)+\lambda\mathcal{K}^*(\mathcal{K}u-u_0)=0, \qquad \text{in } \Omega\times(0,T], \tag{1.3a}$$

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$$\frac{\partial u}{\partial n}=0, \qquad \text{in } \partial\Omega\times(0,T], \tag{1.3b}$$

$$u(0)=u_0, \qquad \text{in } \Omega, \tag{1.3c}$$

where $\mathcal{K}^*$ is the adjoint operator of $\mathcal{K}$, $\psi(x,r): \mathbb{R}^N\times\mathbb{R}^+\to\mathbb{R}^+$ is increasing with respect to r, which directs the regularization behavior and penalizes high gradients of u_0.

As the additive noise η is considered as high frequency variations with low amplitude, the functional $\psi(x,r)=q(x)r^2$ is traditionally selected to remove it. However, with the smoothness of the solution, the edges of pictures are scraped off at the same time. To solve this problem, Rudin, Osher and Fatemi [2] proposed an adaptive total variation model to maintain sharpness of the edges by the functional $\psi(x,r)=q(x)r$, which resulted in the so-called TV-based diffusion and the BV solutions to protect discontinuity. Subsequently, lots of models combining TV-based and isotropic diffusion were established, see, e.g., Acar and Vogel [3], Chambolle and Lions [4], Chan and Shen [5] and the reference therein. Recently, Chen, Levine and Rao [6] introduced an edge detector into the function $\psi(x,r)$ (see the representations (2.5) and (2.6)), which was very good in the edge enhancement, but the deblurring was not considered in their model.

The well-posedness of the minimization problem(1.2) and its corresponding evolutional problem (1.3) in BV space have been first discussed by Vese [7]. However, due to the methods of using maximal monotone operators in Hilbert space, the results of [7] for the evolutional problem (1.3) are only in the case of dimensions $N=1$ and $N=2$. Chen and Wunderli [8] offered another approach to prove the well-posedness of the image denoising problems, which are the problem (1.2) and (1.3) with $\mathcal{K}=\mathrm{I}$ in BV space such that the limit on the dimensions was rid off.

Besides, many techniques have been proposed and widely used as to the numerical implementation of the problem (1.2) and (1.3). One point worth mentioning is the convolution part, which could not be solved by applying traditional matrix multiplication methods due to the high cost to handle such an extremely large system. To solve this problem, Vogel in [9] presented the Fast Fourier Transformation (FFT) methods. Moreover, considering some specific features of blur kernels, some approximation methods such as the local template convolution have also been applied, which could be efficiently used under the MATLAB environment.

In this paper we mainly follow the approach of [8] to investigate the evolutional problem (1.2) and more attention will be paid on the blur operator. We will take approximate methods to prove the well-posedness of the evolution problem (1.3)—a nonlinear integro-differential equation. Besides, we want to get rid of the limitation of $N\leq 2$. Recently Chang and Chern [10] made some numerical experiments on the evolution problem associated with deblurring and denoising by finite differences schemes and found that the discrete blur operator $\mathcal{K}$ makes the algorithm matrix absent diagonal dominance. Their numerical observation motivates us to consider the associated evolution problem by a suitable approximate model with a small viscosity $\epsilon\triangle u$, which is just one kind of versions introduced by Chen and Wunderli in [8].