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Minimal Hypersurfaces in Hyperbolic Spaces

SUN Jun*

Institute of Mathematics, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100190, China.

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Abstract. In this paper, we reprove a theorem of M. Anderson [Invent. Math., 69 (1982), pp. 477-494] which established the existence of a minimal hypersurface in the hyperbolic space with prescribed asymptotic boundary with non-negative mean curvature in the non-parametric case. We use the mean curvature flow method.

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1 Introduction

It is an interesting problem to prove the existence of minimal surfaces in a given Riemannian manifold. In 1982, Anderson [1] studied the minimal varieties in hyperbolic spaces. His main result [1, Theorem 3] says that there exists a complete minimal, absolutely area minimizing locally integral p-current Σ in $\mathbf{H}^n$ asymptotic to M^{p-1} at infinity for any immersed closed submanifold M^{p-1} in the sphere at infinity of $\mathbf{H}^n$. At the end of his paper, Anderson proved the corresponding result for the non-parametric case [1, Theorem 10]. Indeed, he proved the existence and uniqueness of the solution with the boundary condition to the minimal hypersurface equation for the upperhalf space model of $\mathbf{H}^n$

$$\begin{cases} \Delta f - \sum\limits_{i,j=1}^{n-1} \dfrac{f_{x_i} f_{x_j} f_{x_i x_j}}{1+|Df|^2} + \dfrac{n-1}{f} = 0, & \text{in } \Omega, \\ f(x) = 0, & \text{for } x \in \partial\Omega. \end{cases} \tag{1.1}$$

The proof of [1] is mainly based on the convex hull property for the stationary integral p-current which gives the uniform upper and lower bounds on any compact set K in Ω. The proof needs knowledge about geometric measure theory.

*Corresponding author. *Email address:* sunjun@amss.ac.cn (J. Sun)

In this paper, we give another proof of the theorem using the mean curvature flow approach (see Section 4). More precisely, we consider the initial-boundary problem

$$\begin{cases} \dfrac{\mathrm{d}f}{\mathrm{d}t} = \Delta f - \displaystyle\sum_{i,j=1}^{n-1} \frac{f_{x_i} f_{x_j} f_{x_i x_j}}{1+|Df|^2} + \frac{n-1}{f}, & \text{in } \Omega\times[0,\infty), \\ f(x,0) = f_0(x), & \text{for } x\in\Omega, \\ f(x,t) = f_0(x) = 0, & \text{for } x\in\partial\Omega,\ t\in[0,\infty). \end{cases} \tag{1.2}$$

Our basic tool is the comparison principle for the mean curvature flow [2,3]. Using this we can obtain the *a priori* estimates we need. We compare the solution of (1.2) initiated from an arbitrary graph with the hemisphere, which is a stationary solution to the mean curvature flow (Proposition 2.1). The standard parabolic estimates guarantee the global existence and convergence of the mean curvature flow.

Indeed, Huisken [4] has already used the mean curvature flow method to prove the corresponding theorem for the Euclidean case. Of course, maximum principle is enough in his case because of the lack of the term $\frac{n-1}{f}$. On the other hand, the comparison principle is used by Ecker and Huisken when they study the global existence of the mean curvature flow for the graphic case [5].

Our paper is organized as follows: In Section 2, we compute some basic geometric quantities in $\mathbf{H}^n$, and use them to prove that the hemisphere in the upperhalf space is totally geodesic in the hyperbolic space. In Section 3, we establish the equivalence between the mean curvature flow equation and (1.2) and prove the comparison principle for mean curvature flow. Indeed, Huisken proved this theorem in [2]. For the purpose of completeness, we prove it here again. Finally, we prove the main theorem in Section 4.

2 Preliminaries

Let $\mathbf{H}^n$ denote the hyperbolic n-space of constant curvature -1. We will use several models of $\mathbf{H}^n$. For example, we will identify $\mathbf{H}^n$ with the unit ball in $\mathbf{R}^n$ via the Poincaré model. This model exhibits the conformal equivalence of $\mathbf{H}^n$ with $\mathbf{R}^n$, and the hyperbolic metric on $\mathbf{B}^n$ is given by

$$\mathrm{d}s^2_{\mathbf{H}} = \frac{4}{(1-r^2)^2}\mathrm{d}s^2_{\mathbf{E}},$$

where $ds^2_{\mathbf{E}}$ is the Euclidean metric, and r is the distant from the origin. In this model, the sphere $\partial\mathbf{B}^n$ is called *the sphere at infinity* and denoted by $S^{n-1}(\infty)$.

We have the natural compactification of $\mathbf{H}^n$

$$\overline{\mathbf{H}}^n = \mathbf{H}^n \cup S^{n-1}(\infty)$$

given by the Poincaré model. And we define the *asymptotic boundary* S of p-dimensional submanifold Σ in $\mathbf{H}^n$ by

$$S = \overline{\mathrm{supp}\Sigma} \cap S^{n-1}(\infty),$$