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Global Weak Solutions to One-Dimensional Compressible Navier-Stokes Equations with Density-Dependent Viscosity Coefficients

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Abstract. We prove the global existence of weak solutions of the one-dimensional compressible Navier-stokes equations with density-dependent viscosity. In particular, we assume that the initial density belongs to L^1 and L^∞, module constant states at $x=-\infty$ and $x=+\infty$, which may be different. The initial vacuum is permitted in this paper and the results may apply to the one-dimensional Saint-Venant model for shallow water.

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1 Introduction

Consider the one-dimensional (1D) compressible Navier-Stokes equations with density-dependent viscosity coefficients:

$$\rho_t+(\rho u)_x=0, \tag{1.1}$$
$$(\rho u)_t+(\rho u^2+P(\rho))_x=(\mu(\rho)u_x)_x. \tag{1.2}$$

Here $\rho(x,t)$, $u(x,t)$ and $P(\rho)=\rho^\gamma$ $(\gamma>1)$ stand for the fluid density, velocity and pressure respectively. For simplicity, the viscosity coefficient $\mu(\rho)$ is assumed to be $\mu(\rho)=\rho^\alpha$ with $\alpha>\frac{1}{2}$. The initial data is imposed as

$$(\rho,\rho u)|_{t=0}=(\rho_0,m_0)(x). \tag{1.3}$$

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When the viscosity $\mu(\rho)$ is a positive constant, there has been a lot of investigations on the compressible Navier-Stokes equations, for smooth initial data or discontinuous initial data, one-dimensional or multidimensional problems (see [1–5] and the references therein). In [6], Hoff proved the existence of global weak solutions with large discontinuous initial data, possibly having different limits at $x=\pm\infty$. He proved moreover that the constructed solutions have strictly positive densities (vacuum states can't form in finite time). The global existence result in multidimensional case when initial density is allowed to vanish was due to lions (see [4]) and later was improved by E. Feireisl (see [5]). However, the studies in [2, 7, 8] show that the compressible Navier-Stokes equations with constant viscosity coefficients behave singularly in the presence of vacuum. By some physical considerations, Liu, Xin and Yang in [8] introduced the modified compressible Navier-Stokes equations with density-dependent viscosity coefficients for isentropic fluids. As presented in [8], in deriving the compressible Navier-Stokes equations from the Boltzmann equations by the Chapman-Enskog expansions, the viscosity depends on the temperature, and correspondingly depends on the density for isentropic cases. Meanwhile, an one-dimensional viscous Saint-Venant system for shallow water, which is derived rigorously by Gerbeau-Perthame recently (see [9]), is expressed exactly as (1.1)-(1.2) with $\mu(\rho)=\rho$ and $\gamma=2$.

However, few results are available for multi-dimensional problems. The first multidimensional result is due to Bresch, Desjardins and Lin [10], where they showed the L^1 stability of weak solutions for the Korteweg's system (with the Korteweg stress tensor $k\rho\nabla\triangle\rho$) and their result was later improved in [11] to include the case of vanishing capillarity ($k=0$), but with an additional quadratic friction term $r\rho|\mathrm{U}|\mathrm{U}$. An interesting new entropy estimate is established in [10] and [11] in a priori way, which provides some high regularity for the density. Recently, Mellet and Vasseur [12] proved the L^1 stability of weak solutions based on the new entropy estimate, extending the corresponding L^1 stability results of [10] and [11] to the case $r=k=0$. However, although L^1 stability is considered as one of the main steps to prove existence of weak solutions, the global existence of weak solutions of Korteweg's system (see [10]) and the compressible Navier-Stokes equations with density-dependent viscosity seems highly non-trivial in the multi-dimensional cases.

Recently, Jiu and Xin in [13] obtained the global existence of the weak solutions for one-dimensional compressible Navier-Stokes equations with density-dependent viscosity coefficients and furthermore studied the asymptotic behaviors of the weak solutions. The global existence of weak solutions for the initial-boundary-value problems for spherically symmetric compressible Navier-Stokes equations with density-dependent viscosity was proved by Guo, Jiu and Xin in [14]. Initial-boundary-value problems for one-dimensional equations (1.1), (1.2) with $\mu(\rho)=\rho^\alpha$ ($\alpha>\frac{1}{2}$) was studied by Li, Li and Xin (see[15]), the phenomena of vacuum vanishing and blow up of solutions were found there. It is noted that the results in [13–15] are valid for the viscous Saint-Venant system for shallow water.

This paper is concerned with the global existence of the weak solutions to the Cauchy