JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **23**, No. 4, pp. 305-314

doi: 10.4208/jpde.v23.n4.1
November 2010

# On Existence of Ground States for Some Elliptic Systems

WEI Gongming*

*College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China.*

Received 28 August 2007; Accepted 31 August 2010

**Abstract.** In this paper we consider the existence of ground states for some 2- coupled nonlinear Schrödinger systems with or without potentials. Under various conditions on the parameters in the equations, we prove the existence of ground states.

**AMS Subject Classifications**: 35J50

**Chinese Library Classifications**: O175.2

**Key Words**: Ground state; solution manifold; critical point theory; coupled nonlinear Schrödinger system.

## 1 Introduction

In this paper, we consider the existence of ground states of

$$\begin{cases} \Delta u-\lambda_1 u+\mu_1 u^3+\beta uv^2=0, \\ \Delta v-\lambda_2 v+\mu_2 v^3+\beta u^2 v=0, \end{cases} \quad \text{in } \mathbf{R}^n, \tag{1.1}$$

and

$$\begin{cases} \Delta u-V_1(x)u+\mu_1 u^3+\beta uv^2=0, \\ \Delta v-V_2(x)v+\mu_2 v^3+\beta u^2 v=0, \end{cases} \quad \text{in } \mathbf{R}^n, \tag{1.2}$$

where $n=1,2,3$, $\lambda_i$'s are positive constants and $V_i$'s are smooth positive potentials and bounded from below. Problem (1.1) has applications in many physical problems, especially in nonlinear optics. Problem (1.2) arises in the Hartree-Fock theory for a double condensate, i.e. a binary mixture of Bose-Einstein condensates in two different hyperfine

*Corresponding author. *Email address:* `gmweixy@163.com` (G. Wei)

states. The coupling constant $\beta$ describes the interaction between these two states. $\mu_i$'s are for self-focusing or de-focusing of the single states, depending on their signs. Let

$$E_i=\left\{u\in H^1(\mathbf{R}^n)\Big|\int_{\mathbf{R}^n}V_i(x)u^2\mathrm{d}x<\infty\right\}$$

and $E=E_1\times E_2$. In this paper, our assumptions on $V_i(x)$'s are the following:

| (H) $E_i$'s are compactly imbedded into $L^p(\mathbf{R}^n)$ for $2\le p<2^*$. |
|---|

This assumption is a generalization of [1] and [2].

**Theorem 1.1.** *Suppose $\mu_i\le 0$ and $\beta>\sqrt{\mu_1\mu_2}$. Then problem* (1.1) *has ground state.*

**Theorem 1.2.** *Suppose $V_i$'s satisfy assumption* (H) *and one of the following conditions:*
*(i) $\mu_i>0$, $\beta<0$;*
*(ii) $\mu_i>0$, $\beta>\beta_0$ for some positive constant $\beta_0$;*
*(iii) $\mu_i\le 0$, $\beta>\sqrt{\mu_1\mu_2}$.*
*Then problem* (1.2) *has ground state.*

Many authors have studied the existence and multiplicity of solutions of the following Schrödinger equation

$$\Delta u-V(x)u+f(u)=0,\quad x\in\mathbf{R}^n$$

under general assumptions on $V$, see, e.g., [1–10]. Since the milestone work of Floer and Weinstein [11], there have been a lot of works on the existence and concentration of ground states of the singularly perturbed problem

$$h^2\Delta u-V(x)u+f(u)=0,\quad x\in\mathbf{R}^n$$

as $h\to 0+$, see, e.g., [12–15]. Ni [16] provided a comprehensive review on Gierer-Meinhardt type equations and motivated many interesting singularly perturbed problems.

Recently, several authors have considered problems (1.1) and (1.2), including the existence and concentration of ground states, see, e.g., [17–22]. One of the main techniques in their papers is to employ Nehari's solution manifold, which was used in [23] to solve Nehari's problem and developed to deal with Gierer-Meihardt system in [24]. This paper is a continuation and complement of the above results on existence of ground states.

## 2 Proof of Theorem 1.1

*Proof.* Let $H=H^1\times H^1$, where $H^1=H^1(\mathbf{R}^n)$ is the Sobolev space. Define

$$c=\inf_{(u,v)\in N}I(u,v),$$