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Uniqueness of the Weak Extremal Solution to Biharmonic Equation with Logarithmically Convex Nonlinearities

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Abstract. In this note, we investigate the existence of the minimal solution and the uniqueness of the weak extremal (probably singular) solution to the biharmonic equation

$$\triangle^2 \omega = \lambda g(\omega)$$

with Dirichlet boundary condition in the unit ball in $\mathbb{R}^n$, where the source term is logarithmically convex. An example is also given to illustrate that the logarithmical convexity is not a necessary condition to ensure the uniqueness of the extremal solution.

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1 Introduction

In 1960s, a problem, which came from combustion theory [12, 15] and stellar structure [7], had been brought into mathematicians' attention by great Russian mathematician Gel'fand [12]. The problem is written as:

$$\begin{cases} -\triangle u = \lambda e^u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{G}$$

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where Ω is a bounded domain in $\mathbb{R}^n$ $(n\geq 3)$ and $\lambda\geq 0$ is a parameter. It is well known that (see [5,6,8,12,16]) there exists a $\lambda^*>0$, such that:

a) For every $0<\lambda<\lambda^*$, equation (G) has a minimal, positive classical solution u_λ, named minimal solution, which is the unique stable solution of (G);

b) The map $\lambda\mapsto u_\lambda$ is increasing;

c) For $\lambda>\lambda^*$, there is no solution, even in the weak sense;

d) For $\lambda=\lambda^*$, there is a weak solution, called extremal solution, $u^*=\lim_{\lambda\to\lambda^*}u_\lambda$ of (G);

e) The extremal solution u^* is unique, no matter it's singular or regular.

Moreover, the results above are also true for the generalized Gel'fand problem:

$$\begin{cases} -\triangle u=\lambda g(u), & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega, \end{cases} \tag{G_g}$$

where the source term g satisfies the superlinear condition, i.e.

$$\begin{aligned} &g\colon\ [0,+\infty)\mapsto[0,+\infty) \text{ is a } C^1 \text{ convex, nondecreasing function}\\ &\quad\text{with } g(0)>0, \text{ and } \lim_{t\to+\infty}\frac{g(t)}{t}=+\infty. \end{aligned} \tag{C_g1}$$

The appearance of higher order models in physics and mechanics stimulates the study of higher order elliptic equations. Following this trend, Arioli et al. [2] investigated the biharmonic Gel'fand problem:

$$\begin{cases} \triangle^2 u=\lambda e^u, & \text{in } B,\\ u=\dfrac{\partial u}{\partial\nu}=0, & \text{on } \partial B. \end{cases} \tag{G_{bi}}$$

They extended the results a)-d) above to (G_{bi}), while the uniqueness of the extremal solution, i.e. e), was an open problem (see Section 8, [2]). The extension is not absolutely trivial, since the lack of the "maximum principle" which plays such a crucial role in developing the theory for the Laplacian. Indeed, it is well known that such a principe does not normally hold for the general domain Ω (at least for the clamped boundary condition $u=\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$) unless one restricts attention to the unit ball $\Omega=B$ in $\mathbb{R}^n$, where one can exploit a positivity preserving property of $\triangle^2$ due to Boggio [4]. By restricted the domain to the unit ball, the comparison principle, a weak version of maximum principle, is