JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **23**, No. 4, pp. 374-388

doi: 10.4208/jpde.v23.n4.5
November 2010

Solving Inhomogeneous Linear Partial Differential Equations

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Received 12 May 2010; Accepted 21 June 2010

Abstract. Lagrange's variation-of-constants method for solving linear inhomogeneous ordinary differential equations (ode's) is replaced by a method based on the Loewy decomposition of the corresponding homogeneous equation. It uses only properties of the equations and not of its solutions. As a consequence it has the advantage that it may be generalized for partial differential equations (pde's). It is applied to equations of second order in two independent variables, and to a certain system of third-order pde's. Therewith all possible linear inhomogeneous pde's are covered that may occur when third-order linear homogeneous pde's in two independent variables are solved.

AMS Subject Classifications: 35C05, 35G05

Chinese Library Classifications: O175.4

Key Words: Partial differential equations; linear differential equations; inhomogeneous differential equations.

1 Introduction

Linear differential equations have been considered extensively in the mathematical literature, beginning in the second half of the 19th century. For linear homogeneous ordinary differential equations (ode's) there exists a fairly complete theory, culminating in differential Galois theory and algorithms for finding large classes of solutions. Here this means always a closed form solution in some well defined function space; in particular numerical or graphical solutions are excluded. For inhomogeneous equations, Lagrange's method of variation-of-constants allows finding a special solution if a fundamental system for the homogeneous equation is known.

For linear partial differential equations (pde's) the answer is much less complete. For homogeneous equations factorizations and Loewy decompositions appear to be the best

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tool for solving them. However, virtually nothing has been done for solving *inhomogeneous* pde's. The situation is complicated by the fact that it does not seem to be possible to adjust Lagrange's method for pde's.

Therefore in this article a new approach is suggested that does not rely on Lagrange's method. It uses the right divisors that may exist for the differential operator corresponding to the left-hand side of an equation and constructs the inhomogeneities for the lower-order equations corresponding to them. In a second step, a special solution for the originally given equation is generated.

In this way, for reducible second-order equations in two variables the complete answer is obtained in this article. Furthermore, a certain inhomogeneous third-order system is considered; it may occur when third-order linear homogeneous pde's in the plane are considered that are not completely reducible.

In the subsequent Section 2 inhomogeneous ode's are solved by a method that does not apply variation-of-constants. In Section 3 this method is adapted for second-order pde's in two independent variables. Section 4 deals with the above mentioned third-order system.

2 Solving inhomogeneous ODE's

Let an inhomogeneous linear ode for an unknown y depending on x be given by

$$Ly \equiv (D^n + q_1 D^{n-1} + q_2 D^{n-2} + \cdots + q_{n-1} D + q_n) y = R, \tag{2.1}$$

where $D \equiv \frac{d}{dx}$ and $R \neq 0$. Let $\{y_1, \cdots, y_n\}$ be a fundamental system for (2.1). Then the general solution of the homogeneous equation $Ly = 0$ is $y = C_1 y_1 + \cdots + C_n y_n$; the C_i are constants. Solving the inhomogeneous equation means to find a special solution y_0 such that $Ly_0 = R$.

The classical procedure for determining y_0 is the method of *variation of constants* [1], Section 5.23, page 122; originally it is due to Lagrange. According to this method, the C_i in the general solution of the homogeneous equation are considered as functions of x. Substituting the resulting expression into (2.1) and imposing the $n-1$ constraints

$$C_1' y_1^{(k)} + C_2' y_2^{(k)} + \cdots + C_n' y_n^{(k)} = 0 \tag{2.2}$$

for $k = 0, \cdots, n-2$ yields the additional condition

$$C_1' y_1^{(n-1)} + C_2' y_2^{(n-1)} + \cdots + C_n' y_n^{(n-1)} = R. \tag{2.3}$$

The linear algebraic system (2.2), (2.3) for the C_k' has always a solution due to the non-vanishing determinant of its coefficient matrix which is the Wronskian $W^{(n)}$. Consequently determining a special solution y_0 of the inhomogeneous equation (2.1) requires only integrations if a fundamental system for the corresponding homogeneous problem