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Logarithmic Gradient Estimates to Hessian-Type Equations

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Abstract. We study logarithmic gradient estimate to smooth admissible solutions of the Hessian-type equations on $\mathbb{S}^n$. The equations contain a Monge-Ampère equation arising in designing a reflecting surface in geometric optics as a special case.

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1 Introduction

In this paper, we give logarithmic gradient estimate to the admissible solution of the following equations

$$\sigma_k(K_{ij})=f, \qquad \text{on } \mathbb{S}^n, \tag{1.1}$$

where

$$K_{ij}=\frac{2uu_{ij}+(u^2-|\nabla u|^2)\delta_{ij}}{u^2+|\nabla u|^2}, \tag{1.2}$$

and σ_k is the k-th elementary symmetric function (see Definition 2.1 below).

K_{ij} is referred to as intensity density in [1]. The case for $k=n$ of equation (1.1) is an equation of Monge-Ampère type arising in geometric optics. Let Γ be a reflecting hyper-surface in $\mathbb{R}^{n+1}$ which is a graph over some domain $\phi\subset\mathbb{S}^n$. Denote by $x=x(s)\equiv x(s^1,\cdots,s^n)$ a smooth parametrization of ϕ in local coordinates. Such Γ can be defined by the position vector $r(x)=\rho(x)x$, where $x\in\phi$ and ρ is the radial function of Γ, which is positive and C^2 on ϕ. If we identify each direction of the ray with a point on $\mathbb{S}^n$, and the

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ray of the light reflects according to geometric optics, then the direction of the reflected light defines another point on $\mathbb{S}^n$. Hence, we obtain a map from $\mathbb{S}^n$ to $\mathbb{S}^n$. Let $N(x)$ denote unit outer normal of Γ at $\rho(x)x$, and $y=T(x)=T_\rho(x)$ the direction of the light reflected by Γ. If we regard a unit vector as a point on $\mathbb{S}^n$ and by the reflection law we have

$$y=T_\rho(x)=x-2\langle x,N(x)\rangle N(x). \tag{1.3}$$

Suppose no energy is lost in reflection, and by the energy conservation law, the Jacobian of $T(x)$ is equal to $\frac{f(x)}{g(T(x))}$, which leads to the equation

$$\frac{\det(\nabla_{ij}u+(u-\eta)e_{ij})}{\eta^n\det(e_{ij})}=\frac{f(x)}{g(Tx)}.$$

Where $u=\frac{1}{\rho}$, ∇_{ij} denotes the covariant derivatives on $\mathbb{S}^n$, e is the standard metric on $\mathbb{S}^n$ induced from $\mathbb{R}^{n+1}$, and

$$\eta=\frac{(u^2+|\nabla u|^2)}{2u}.$$

$f(x)$ denotes the intensity of the source O and $g(y)$ the distribution of the directions of the reflected light on $\mathbb{S}^n$, the detailed derivation of the equation can be seen in [2] and the references therein. In general case for σ_k, there is a derivation of these Hessian-type equations in [1], where K_{ij} is similar to the second fundamental form of the surface.

There are Hessian equations arising from conformal geometry which are similar to problems (1.1) and (1.2), and there have been some works concerning global and local gradient estimates. In [3], Guan and Wang obtained the gradient estimate of admissible solution to the σ_k of eigenvalues of Schouten tensors on compact Riemannian manifolds. In [4], the local gradient estimate was extended to the case of conformal quotient equation of Schouten tensors on complete Riemannian manifolds.

For $k=1$, Eq. (1.1) is of quasi-linear type. Oliker obtained gradient estimate in [1] and got the existence and uniqueness. For $k=n$, Eq. (1.1) is of Monge-Ampére type. In this case, the existence, uniqueness, and smoothness of the reflecting surface are studied by Wang in [5]. Guan and Wang in [2] obtained the existence and uniqueness of smooth solutions up to multiplication of positive constants under the condition

$$\int_{\mathbb{S}^n}f(x)\mathrm{d}x=\int_{\mathbb{S}^n}g(x)\mathrm{d}x.$$

In [6], Wang considered the existence of admissible solutions to a class of fully nonlinear equations, with the Conformal k-Hessian equation as a special case. In deriving the first-derivative estimates, Wang used a blow up argument and the Liouville theorem in [7].

It is well-known that Yau [8] gave logarithmic gradient estimate of positive solution to harmonic function on complete Riemannian manifold. As applications, he proved Liouville theorem on complete Riemannian manifold with nonnegative Ricci curvature. Logarithmic gradient estimates also can be used to derive Harnack inequality through path integral. In this paper, we consider the logarithmic gradient estimates to this Hessian type equation. Our result is as follows.