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Decay of Solutions to a 2D Schrödinger Equation

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Abstract. Let $u\in C(\mathbb{R},H^1)$ be the solution to the initial value problem for a 2D semi-linear Schrödinger equation with exponential type nonlinearity, given in [1]. We prove that the L^r norms of u decay as $t\to\pm\infty$, provided that $r>2$.

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1 Introduction

In this work, we study some asymptotic properties of solution to the following initial value Schrödinger equation

$$i\partial_t u+\Delta_x u=f(u), \qquad \text{in } \mathbb{R}_t\times\mathbb{R}^2_x, \tag{1.1}$$

with data

$$u_0:=u(0,.)\in H^1(\mathbb{R}^2), \tag{1.2}$$

where $u:=u(t,x)$ is a complex-valued function of $(t,x)\in\mathbb{R}\times\mathbb{R}^2$, and

$$f(u):=u\left(e^{4\pi|u|^2}-1\right). \tag{1.3}$$

Two important conserved quantities of (1.1) are the mass and the Hamiltonian. The mass is defined by

$$M(u(t)):=\|u(t)\|^2_{L^2(\mathbb{R}^2)}, \tag{1.4}$$

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and the Hamiltonian is defined by

$$H(u(t)):=\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)}+\frac{1}{4\pi}\|e^{4\pi|u(t)|^2}-1-4\pi|u(t)|^2\|_{L^1(\mathbb{R}^2)}. \tag{1.5}$$

We know [1] that the Cauchy problem (1.1)-(1.2) has a unique solution u in the space $C(\mathbb{R},H^1(\mathbb{R}^2))\cap L^4_{loc}(C^{1/2}(\mathbb{R}^2))$. Moreover, u satisfies conservation of the mass and the Hamiltonian. Our aim, in this paper, is to prove some asymptotic properties of such solution.

Before going further, let recall some historic facts about well-posedness of the monomial defocusing semilinear Schrödinger equation

$$i\partial_t u+\Delta_x u=|u|^{p-1}u, \quad p>1, \quad u:(-T^*,T^*)\times\mathbb{R}^d\to\mathbb{C}. \tag{1.6}$$

A solution u to (1.6) satisfies conservation of the mass and the Hamiltonian

$$H_p(u(t)):=\|\nabla u(t)\|^2_{L^2(\mathbb{R}^2)}+\frac{2}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}(t,x)\mathrm{d}x.$$

Moreover, for any $\lambda>0$,

$$\begin{aligned}&u_\lambda:(-T^*\lambda^2,T^*\lambda^2)\times\mathbb{R}^d\to\mathbb{C},\\&(t,x)\longmapsto\lambda^{\frac{2}{1-p}}u(\lambda^{-2}t,\lambda^{-1}x)\end{aligned}$$

is a solution to (1.6). Note also that for $s_c:=d/2-2/(p-1)$, the $\dot{H}^{s_c}(\mathbb{R}^d)$ norm is relevant in the well-posedness theory of (1.6) because it is invariant under the mapping

$$f(x)\longmapsto\lambda^{\frac{2}{1-p}}f(\lambda^{-1}x), \qquad \lambda>0.$$

We refer to Eq. (1.6) with the notation $NLS_p(\mathbb{R}^d)$ and we limit our discussion to the case $0\le s_c\le 1$. If $s_c>1$, (1.6) is locally well-posed in H^s, for $s>s_c$.

1. $NLS_p(\mathbb{R}^d)$ **local well-posedness in** $H^s(\mathbb{R}^d)$. It is known (see, e.g., [2–4]) that

(a) If $s>s_c$, then (1.6) is locally well-posed in H^s, with an existence interval depending only upon $\|u_0\|_{H^s}$.

(b) For $s=s_c$, (1.6) is locally well-posed in H^s, with an existence interval depending upon $e^{it\Delta}u_0$.

(c) If $s<s_c$, then (1.6) is ill-posed in H^s (see, e.g., [5–9]).

So, it is naturel to refer to H^{s_c} as the critical regularity for (1.6). 2. $NLS_p(\mathbb{R}^d)$ **global well-posedness** .