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Regularity Theorems for Elliptic and Hypoelliptic Operators via the Global Relation

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Abstract. In this paper we give a new proof regarding the regularity of solutions to hypoelliptic partial differential equations with constant coefficients. On the assumption of existence, we provide a spectral representation for the solution and use this spectral representation to deduce regularity results. By exploiting analyticity properties of the terms within the spectral representation, we are able to give simple estimates for the size of the derivatives of the solutions and interpret them in terms of Gevrey classes.

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1 Introduction

Studies concerning the regularity of solutions to partial differential equations (PDEs) with constant coefficients have a long and rich history. Much of this work originated from interest in the smoothness of solutions to the Laplace's equation, for which Weyl [1] showed that all weak solutions are in fact smooth solutions using his method of orthogonal projection. Regarding higher regularity conditions, Petrowsky [2] proved the analyticity of classical solutions to homogeneous, elliptic PDEs with constant coefficients. One of the most beautiful results in the study of regularity of solutions to constant coefficient PDEs is due to Hörmander [3], who gave a complete algebraic classification of operators $P(D)$ which have the hypoelliptic property:

$$P(D)u \in C^\infty \Rightarrow u \in C^\infty.$$

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We refer the reader to [4–7] for an in depth account of this result, whereas a less detailed but more accessible discussion can be found in [8].

Hörmander's theorem theorem tells us that if $P \in \mathbf{C}[\lambda]$ is a polynomial in $\lambda \in \mathbf{C}^n$ such that $|\mathrm{Im}\lambda| \to \infty$ as $\lambda \to \infty$ in the algebraic variety Z_P:

$$Z_P = \{\lambda \in \mathbf{C}^n : P(\lambda) = 0\},$$

then $P(D)$ with $D = -\mathrm{i}\partial$ is hypoelliptic, i.e. $\mathrm{sing\,supp}\, u = \mathrm{sing\,supp}\, Pu$. Hörmander's original proof relies on the construction of a parametrix for the operator $P(D)$ and deriving properties of this parametrix using very precise results regarding the variety Z_P and the growth of the functions $P(\lambda)^{-1} D^\alpha P(\lambda)$. In this paper, we give an alternate proof of the theorem. Using some relatively low-tech methods, we provide a spectral representation of the function u which satisfies

$$P(D)u = f, \qquad \text{in } \Omega \subset \mathbf{R}^n,$$

from which the regularity of u can readily be seen. The theorem is precisely stated as follows.

Theorem 1.1. *Let $P \in \mathbf{C}[\lambda]$ be such that*

$$|\mathrm{Im}\lambda| \to \infty, \qquad \text{as } \lambda \to \infty$$

in the variety Z_P. Suppose $u \in \mathcal{D}'(\Omega)$ satisfies $P(D)u = f$ in the distributional sense. Then P is hypoelliptic and for each $x \in \Omega$, u has a spectral representation of the form

$$u(x) = E * f(x) + \sum_I \int_{\Sigma_I} e^{\mathrm{i}\lambda \cdot x} U_I(\lambda)\, \mathrm{d}\lambda, \tag{1.1}$$

where $E \in \mathcal{S}'(\mathbf{R}^n)$ is a known function given in terns of $P(\lambda)$ and the finite collection of known surfaces $\Sigma_I \subset \mathbf{C}^n$ are such that the known functions $e^{\mathrm{i}\lambda \cdot x} U_I(\lambda)$ have rapid decay, i.e.,

$$e^{\mathrm{i}\lambda \cdot x} U_I(\lambda) \in \mathcal{S}(\Sigma_I),$$

for fixed $x \in \Omega$. The derivatives of u are given by

$$D^\alpha u(x) = E * D^\alpha f(x) + \sum_I \int_{\Sigma_I} \lambda^\alpha e^{\mathrm{i}\lambda \cdot x} U_I(\lambda)\, \mathrm{d}\lambda,$$

for each fixed $x \in \Omega$.

The spectral representation (1.1) is explicit. Both E and the $\{U_I\}$ are known, and are expressible in terms of the distributions $u, f \in \mathcal{D}'(\Omega)$. The exact description of these objects is given in Proposition 2.1. Clearly the function $E \in \mathcal{S}'(\mathbf{R}^n)$ represents a parametrix for $P(D)$, since (1.1) implies:

$$C^\infty \ni P(D)E * f - P(D)u = (P(D)E - \delta) * f,$$