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A Note about Parabolic Systems and Analytic Semigroups

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Abstract. We investigate the question whether certain parabolic systems in the sense of Petrovskii fulfill the resolvent estimate required for the generation of an analytic semigroup and apply the result to a problem concerning the diffusion of gases.

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1 Introduction

Due to their importance in describing, e.g., reaction-diffusion processes and ecological phenomena, strongly coupled systems of nonlinear parabolic differential equations have received much attention (see, among many others [1–8]). Although these equations are often non-linear, the theory of linear parabolic systems is interesting in its own right as well as essential for the non-linear theory. In general one can treat the time as a variable not entirely unlike the space variables. This point of view is pursued in [6,7,9,10]. As an alternative one can see a parabolic system as a rather singular ordinary differential equation, the approach of, e.g. [8,11,12]. The latter approach is to a significant part dependent on the use of analytic semigroups and is often very abstract. In all of these papers the question of local existence is addressed by considering the linearization of the equation itself-or its principal part-which can be written in the abstract form $U_t+L(t)U=f$ with a family of linear operators $L(t)$. Amann [1] gives the condition of strong α-regular ellipticity which guarantees that the operators $-L(t)$ generate analytic semigroups and fulfill certain additional a-priori estimates, probably the most general one under which the method of Agmon [13] still gives the necessary estimates for the resolvent $(L(t)+zI)^{-1}$.

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All systems that are uniformly strongly elliptic and all so called block-upper triangular systems are, e.g., α-regularly elliptic. For systems of second order we derive, also using Agmon's idea, a condition which is more explicit than strong α-regular ellipticity, more general than block-upper triangularity, and actually coincides largely with the definition of parabolicity given in Petrovskii [10] for the linearized differential equation itself. For the boundary condition we use the non-algebraic version of the complementing condition of [14]. We then apply our result to the linearization of the system composed of the equations for the thermal diffusion and mass transfer for a mixture of two gases

$$\begin{aligned} c_t &= \operatorname{div}\left(D\nabla c + \frac{Dk_T}{T}\nabla T\right) \\ T_t &= \operatorname{div}\left(\left[\chi + \frac{(k_T)^2 \mu_c D}{c_\rho T}\right]\nabla T + \frac{Dk_T \mu_c}{c_\rho}\nabla c\right), \end{aligned}$$

which is neither strongly elliptic nor block-upper triangular. Here T is the temperature, c the relative concentration of one of the gases, D, k_T, c and c_ρ are functions of T and c. (See equations of [15, 59.14 and 59.15 on p. 233] which give a version of this system with constant coefficients.) These results are not strictly new, but it still seems worthwhile to publish a simple proof of these theorems.

Let $\beta \in (0,1)$ and $s \in \{0,1,\cdots\}$. Then let Ω be an open connected subset of $\mathbb{R}^n$ $(n \geq 2)$, and let $\partial\Omega \in C^{2+\beta+s}$ be compact. We consider operators of order two of the form

$$(Lu)^l = \sum_{k=1}^{m}\sum_{i,j=1}^{n} a_k^{ijl}(x)\frac{\partial^2 u^k}{\partial x_i \partial x_j}(x)$$

with $l = 1,\cdots,m$, $a_k^{ijl} \in C^{\beta+s}(\overline{\Omega})$ and $a_k^{ijl} = a_k^{jil}$, and with the boundary conditions

$$B^l u = \sum_{i=1}^{n}\sum_{k=1}^{m} b_k^{il}(x)\frac{\partial u^k}{\partial x_i} + \sum_{k=1}^{m} c_k^l(x) u^k = 0,$$

where $b^l = [b_k^{il}(x)]$ either identically equals zero or is unequal to zero on all of $\partial\Omega$ $(l = 1,\cdots,m)$. We have to assume $b \in C^{1+s+\beta}(\partial\Omega)$ and $c = [c_k^l] \in C^{2+s+\beta}(\partial\Omega)$. If L and the boundary conditions fulfill the following three conditions, then we can prove that $-L$ generates an analytic L_p-semigroup for $p \in (1,\infty)$, thus allowing us to use the semigroup methods one can find, e.g., in [11]. In what follows we use the Einstein summation convention.

To formulate the three conditions let

$$A_k^l(x,\pi) = a_k^{ijl}(x)\pi_i\pi_j$$

for $x \in \overline{\Omega}, \pi \in \mathbb{R}^n$, so $Lu = A(x,D)u$. For $l = 1,..,m$ let $[c_k^{*l}]$ equal $[c_k^l]$ if $b^l = 0$, otherwise be zero. If $b^l = 0$ let $q_l = 0$, otherwise $q_l = 1$. This defines the principal part of our boundary condition and the order of each component. Also let

$$D(L) = \{u \in H_p^2(\Omega) \,|\, B^l u = 0 \quad \text{for } l = 1,\cdots,m\}. \tag{1.1}$$