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Nonexistence of Smooth Axially Symmetric Harmonic Maps from B^3 into S^2

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Abstract. Inspired by the construction of blow-up solutions of the heat flow of harmonic maps from D^2 into S^2 via maximum principle (Chang et al., J. Diff. Geom., 36, 1992, pp. 507-515.) we provide examples of nonexistence of smooth axially symmetric harmonic maps from B^3 into S^2 with smooth boundary maps of degree zero.

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1 Introduction

Let B^3 be the unit ball in $\mathbb{R}^3$ and $S^2=\partial B^3$ the unit 2-sphere. Given a smooth map $\varphi: S^2\to S^2$ (we assume φ is at least $C^{1,\alpha}$), φ can always extend to a weakly harmonic map $u_0\in H^1 B^3,S^2)$ which minimizes the energy

$$E(u)=\frac{1}{2}\int_{B^3}|Du|^2\mathrm{d}x$$

on $H^1_\varphi(B^3,S^2)=\{u\in H^1(B^3,S^2): u|_{\partial B^3}=\varphi\}$. In general, u_0 is not smooth and can have isolated singularities (see [2,3]). In fact, a necessary condition for φ to have a continuous extension in B^3 is that $\deg(\varphi)$, the Brower degree of φ, equals 0. An interesting question which has attracted many authors' attention is as follows.

(Q)*: Suppose that* $\deg(\varphi)=0$*. Does there exist a smooth harmonic extension of* φ*?*

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Despite many serious efforts (cf. [4–7]), this question is still open. It is the discovery of the so-called "gap phenomenon" by R. Hardt and F. H. Lin [8] that brought for the first time (Q) into people's attention. They showed that there exist boundary maps φ such that

$$\min_{H^1_\varphi(B^3,S^2)} E(u) < \inf_{H^1_\varphi\cap C^0(B^3,S^2)} E(u).$$

This implies that one cannot solve (Q) by minimizing the energy over H^1_φ. On the other hand, minimization over the restricted class $H^1_\varphi \cap C^0$ has not led to the desired positive answer to (Q). Indeed, it is our belief that the answer to (Q) should be negative in general.

There is an "axially symmetric" version of (Q). A map $u: B^3 \to S^2$ is called axially symmetric, if in cylindrical coordinates,

$$u(r,\theta,t) = (\cos\theta\sin h, \sin\theta\sin h, \cos h), \tag{1.1}$$

for some function $h(r,t)$ defined on the half-disk $\overline{D} = \{(r,t): 0 \le r^2+t^2 \le 1, r \ge 0\}$. One can explain the axially symmetry of u as follows. Let (x_1,x_2,t) be the Euclidean coordinates of $\mathbb{R}^3$, and let R_θ be the rotation about the t-axis through an angle θ. Then u is axially symmetric iff it maps the plane $\{x_2=0\}$ into itself and satisfies

$$u\circ R_\theta = R_\theta \circ u, \quad \forall\theta. \tag{1.2}$$

It is easy to see that, if u is continuous then it has to map the t-axis into either the North pole or the South pole of S^2. We will always assume the former, i.e., u maps the t-axis into the North pole. One can also generalize the condition (1.2) to $u\circ R_\theta = R_{k\theta}\circ u$ for an arbitrary integer k. But that will not lead to more insight for the concerned analytical problem, and we restrict us to the case $k=1$. Note that (1.2) also defines the axially symmetry for the boundary maps φ: $S^2 \to S^2$. Thus we can formulate the axially symmetric version of (Q) as follows.

(Q_s): *Suppose that φ is axially symmetric with deg$(\varphi)=0$. Does there exist a smooth, axially symmetric harmonic extension of φ?*

It is the aim of this note to show that the answer to (Q_s) is generally negative.

For a smooth axially symmetric map $\varphi: S^2 \to S^2$ of degree 0, it is uniquely determined by a smooth function $h_0(r,t)$ defined on $C = \{(r,t): r^2+t^2=1, r\ge 0\} \subset \partial D$, which extends to a smooth function on the whole circle $\{r^2+t^2=1\}$ and satisfies

$$h_0(0,1) = h_0(0,-1) = 0. \tag{1.3}$$

The problem of finding axially symmetric harmonic maps with boundary value φ is then reduced to the following Dirichlet problem on D:

$$h_{tt} + h_{rr} + \frac{1}{r}h_r - \frac{\sin 2h}{2r^2} = 0, \quad \text{in } D, \tag{1.4a}$$
$$h_{|C} = h_0, \tag{1.4b}$$