JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **24**, No. 4, pp. 297-312

doi: 10.4208/jpde.v24.n4.2
November 2011

Analysis of a Free Boundary Problem Modeling Multi-Layer Tumor Growth in Presence of Inhibitor

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Received 10 September 2008; Accepted 16 August 2011

Abstract. In this paper we study well-posedness and asymptotic behavior of solution of a free boundary problem modeling the growth of multi-layer tumors under the action of an external inhibitor. We first prove that this problem is locally well-posed in little Hölder spaces. Next we investigate asymptotic behavior of the solution. By making delicate analysis of spectrum of the linearization of the stationary free boundary problem and using the linearized stability theorem, we prove that if the surface tension coefficient γ is larger than $\gamma^* > 0$ the flat stationary solution is asymptotically stable provided that the constant c representing the ratio between the nutrient diffusion time and the tumor-cell doubling time is sufficient small.

AMS Subject Classifications: 35B35, 35R35, 76D27

Chinese Library Classifications: O357.1

Key Words: Free boundary problem; multi-layer tumor; inhibitor; well-posedness.

1 Introduction

In this paper we consider the following free boundary problem:

$$c\partial_t\sigma = \triangle\sigma - \lambda_1\sigma - \beta \qquad \text{in } \Omega_\rho(t),\ t>0, \tag{1.1}$$
$$c\partial_t\beta = \triangle\beta - \lambda_2\beta \qquad \text{in } \Omega_\rho(t),\ t>0, \tag{1.2}$$
$$\triangle p = -\mu(\sigma - \tilde{\sigma} - \iota\beta) \qquad \text{in } \Omega_\rho(t),\ t>0, \tag{1.3}$$
$$\partial_t\rho = -\nabla p\cdot\vec{n} \qquad \text{on } \Gamma_\rho(t),\ t>0, \tag{1.4}$$
$$\sigma = \bar{\sigma}, \quad \beta = \bar{\beta} \qquad \text{on } \Gamma_\rho(t),\ t>0, \tag{1.5}$$
$$p = \gamma k \qquad \text{on } \Gamma_\rho(t),\ t>0, \tag{1.6}$$

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$$\partial_y\sigma=0,\quad \partial_y\beta=0,\quad \partial_y p=0 \qquad \text{on } \Gamma_0,\ t>0, \tag{1.7}$$

$$\rho(\cdot,0)=\rho_0,\quad \sigma(\cdot,0)=\sigma_0,\quad \beta(\cdot,0)=\beta_0 \qquad \text{at } t=0. \tag{1.8}$$

Here $\sigma=\sigma(x,y,t)$, $\beta=\beta(x,y,t)$ and $p=p(x,y,t)$ respectively represent the concentration of nutrient, concentration of inhibitor and the tumor tissue pressure, λ_1, λ_2, ι, μ and γ are positive constants, c representing the ratio between the nutrient,inhibitor diffusion time and the tumor-cell doubling time is sufficient small, $\vec{n}$ is the unit upward normal field on $\Gamma_\rho(t)$, and κ is the mean curvature of $\Gamma_\rho(t)$. Besides, ρ_0 and Ω_0 are the initial data of $\rho(x,t)$ and $\Omega(t)$, respectively. In the model the tumor is supposed to occupy a n-dimensional region of the form

$$\Omega(t)=\Omega_\rho(t)=\left\{(x,y)\in R^{n-1}\times R\colon\ 0<y<\rho(x,t)\right\}.$$

$\sigma=\bar{\sigma}$ and $\beta=\bar{\beta}$ mean the tumor receives through the upper boundary

$$\Gamma(t)=\Gamma\rho(t)=\left\{(x,y)\in R^{n-1}\times R\colon\ y=\rho(x,t)\right\}$$

constantly supplied nutrient and inhibitors, which diffuses into all parts of the tumor, supporting tumor cells to live and proliferate. The lower boundary $\Gamma_0=\{(x,y)\in R^{n-1}\times R\colon y=0\}$ is fixed. The condition $\partial_y\sigma=0$, $\partial_y\beta=0$, and $\partial_y p=0$ mean the lower boundary is impermeable, so that none of nutrient, inhibitor and tumor cells can pass through it. Such a tumor model describes certain tissue cultures recently developed by medico-biologists, cf. [1–3]. For more explanations of this model, we refer the reader to see [4–7].

For the degenerate case $c=0$, Cui and Escher (cf. [4]) considered the inhibitor-free case (i.e., $\beta\equiv 0$) under the assumption that all unknown functions are periodic in the x-variable. They proved that

(1) the problem is locally well-posed in little Hölder spaces,

(2) there exists a unique flat stationary solution, and

(3) there exists a threshold value $\gamma^*>0$ for the surface tension coefficient γ such that this flat stationary solution is asymptotically stable if $\gamma>\gamma^*$ and unstable otherwise.

These results have been extended in [3] to the case that an inhibitor exists (i.e., $\beta\not\equiv 0$) and all unknown functions are periodic in the x-variable. Naturally, we also want to know whether or not such results also hold for the case that the unknown functions are not necessarily periodic in x. We note that non-periodic case is clearly more difficult than the periodic case, because for the periodic case the above problem is essentially a problem on a compact manifold, so that the embedding operators in the Sobolev-type embedding are generally compact operators, and, consequently, the spectrums of the corresponding linearized operators consist merely of discrete eigenvalues. For the non-periodic case, however, the domain of the problem is non-compact, and the embedding operators in