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Nonradial Solutions of a Mixed Concave-Convex Elliptic Problem

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Abstract. We study the homogeneous Dirichlet problem in a ball for semi-linear elliptic problems derived from the Brezis-Nirenberg one with concave-convex nonlinearities. We are interested in determining non-radial solutions which are invariant with respect to some subgroup of the orthogonal group. We prove that unlike separated nonlinearities, there are two types of solutions, one converging to zero and one diverging. We conclude at the end on the classification of non radial solutions related to the nonlinearity used.

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1 Introduction

The paper is devoted to the study of group $O(n)$ action on the solutions of some mixed concave-convex elliptic equations derived from the famous problem due to Brezis-Nirenberg [1]. In [2,3] the authors focused on the problem

$$\begin{cases} \Delta u+|u|^{p-1}u+\lambda|u|^{q-1}u=0 & \text{in } \Omega_R, \\ u=0 & \text{on } \partial\Omega_R, \end{cases} \tag{1.1}$$

and some derived forms. Ω_R is an open ball of radius R in $\mathbb{R}^n$ and $0<q<1<p$ depending on the Sobolev critical exponent $p_c=(n+2)/(n-2)$. We studied the existence of solutions

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and we have analysed in all subcritical, critical and supercritical cases the possible singularities of radial solutions at the origin. Some numerical methods are also developed to approximate the solution of associated evolutionary nonlinear problems such as

$$\begin{cases} \frac{\partial u}{\partial t}=\Delta u+f(u), & \text{on } \Omega\times[0,T], \\ u(x,t)=0, & \text{on } \partial\Omega\times[0,T], \\ u(x,0)=u_0(x), \end{cases} \tag{1.2}$$

where f is a continuous but non sufficiently regular real valued function looking like $|u|^{p-1}u+\lambda|u|^{q-1}u$. A difference scheme method and a wavelet representation are performed to approximate the solution of the problem above. Remark here that the solution of problem (1.1) can be understood as a time-independent solution of problem (1.2) for suitable $f(u)$. The nonlinear term f is often assumed to be locally Lipschitz continuous which does not hold for many interesting cases in physics such as nonlinear waves, Shrödinger nonlinear equation when dealing with complex case, chemical reaction models, population genetics problems, reactor dynamics and heat transfert. For backgrounds on (1.2), we refer to [4] and the references therein. In [5] and [6], a complete study of nodal solutions has been provided. Compared to the original Brezis-Nirenberg equation, the convex term u^p is replaced by its odd extension $|u|^{p-1}u$ and the linear term disappears completely to be replaced by a second nonlinear non locally lipschitzian odd term $|u|^{q-1}u$.

A natural remark that can be noticed is that any radial solution is invariant under the orthogonal group $O(n)$ action. This brings a second question: Given a subgroup G of $O(n)$, what can be the effect of the G-action on the solutions. An immediate answer is when the group G acts transitively on the sphere $\mathcal{S}^{n-1}$. It is clear in this case that any G-invariant solution can be seen as radial. So, one looks for suitable conditions on G for the existence of non radially symmetric but G-invariant solutions. We mean by G-invariant solution, any solution u of problem (1.1) satisfying

$$u(gx)=u(x), \qquad \forall\, g\in G \quad \text{and} \quad \forall\, x\in\Omega. \tag{1.3}$$

Such question has been already studied by R. Kajikiya in [7] and [8] where the author focused on group invariant solutions of problem (1.1) with $f(u)=|u|^{p-1}u$ and G a closed subgroup of $O(n)$. In fact, when G is not necessarily a closed subgroup but only a subset of $O(n)$, one considers the group generated by it and its closure $\overline{<G>}$. Then G-invariance is equivalent to $\overline{<G>}$ one. This is already mentioned in [7, Remark 2.1].

In the present paper, we suppose as usual that G is a closed subgroup of $O(n)$ and we propose to establish some results about the existence of G-invariant solutions in some more general nonlinearities. The paper is organized as follows. Section 2 is devoted to the statement of main results. In Section 3, the proofs of such results are developed. Next, preliminary results and contexts related to the present work are recalled in an appendix section. A suggested problem induced from the present work is proposed at the