Non-Existence of Global Solutions for a Fractional Wave-Diffusion Equation

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Abstract. We considered the Cauchy problem for the fractional wave-diffusion equation

\[
D^\alpha u - \Delta |u|^{m-1}u + (-\Delta)^{\beta/2}D^\gamma |u|^{l-1}u = h(x,t)|u|^p + f(x,t)
\]

with given initial data and where \(p > 1\), \(1 < \alpha < 2\), \(0 < \beta < 2\), \(0 < \gamma < 1\). Nonexistence results and necessary conditions for global existence are established by means of the test function method. This results extend previous works.

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1 Introduction

In [1], Kirane and Tatar consider the Cauchy problem of the hyperbolic fractional equation

\[
u_{tt} - \Delta u + D^\beta u = h(x,t)|u|^p,
\]

where \(p > 1\) and \(0 < \beta < 1\), this equation arises in the modeling of fast wave propagation in micro-inhomogeneous media see (see [2]). In [1], the authors established conditions on the initial data and the function \(h(x,t)\) that are necessary for local and global existence. It is shown that if

\[
1 < p \leq 1 + \frac{2\beta + \rho}{2 + N - 2\beta},
\]

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(where \( \rho \) comes from the function \( h \)) then we have non-existence of global solutions.

When \( m = l = 1, \alpha = 2, \beta = 0, h = 1 \) and \( \gamma = 1 \), this problem has been treated by a large number of researchers. Then we obtain the wave equation with the linear damping \( u_t \). In this case Todorova and Yordanov [3], Mitidieri and Pohozaev [4] and Zhang [5] showed that the Fujita exponent is \( p_c = 1 + 2/N \). This result has been extended to solutions of the telegraph equation

\[
D^{2\beta}u - \Delta u + D^\beta u = 0,
\]

by Cascaval et al. [6] this problem arises while studying some iterated Brownian motions (see [7]). We point out here that fractional derivatives serve, among other things, to model various anomalous damping such as noise attenuation and viscoelastic dissipations (see [8–12]). Indeed it has been shown by experiments (see [13]) that experiment data fit very well in the models involving fractional derivatives within a broad frequency range for several materials. This materials include synthetic polymers, electrochemistry, glassy materials and many other viscoelastic and hereditary mechanics.

In this paper, we consider the problem

\[
\begin{aligned}
D^\alpha u - \Delta|u|^{m-1}u + (-\Delta)^{\beta/2}D^\gamma |u|^{l-1}u &= h(x,t)|u|^p + f(x,t), \\
\quad u(x,0) &= u_0(x) \geq 0, \quad u_t(x,0) = u_1(x) \geq 0, \quad x \in \mathbb{R}^N.
\end{aligned}
\]

We will generalize the results in [1] to problem (1.2) where \( 1 < \alpha \leq 2, \, 0 < \beta < 2 \) and \( 0 < \gamma < 1 \). Nonexistence results as well as necessary conditions for local and global existence will be established. In addition to this we can look at the equation in problem (1.2) as a generalization of the fractional diffusion-wave equation

\[
D^\alpha u = \Delta u + h(x,t)|u|^p, \quad 1 < \alpha \leq 2.
\]

This eq. (1.3) is now a special case of the eq. (1.2), we can consider the eq. (1.2) as the fractionally damped equation of (1.3). Eq. (1.3) serves as a model in the study of the thermal diffusion in fractal media. See Saichev and Zaslavsky [14], Mainardi [10, 11], Fujita [15] and references therein. Molz et al. in [16] discuss a physical interpretation of the fractional derivative in a Levy diffusion process. Our argument is based on the test-function method developed by Mitidieri and Pohozaev [4], Zhang [5] Kirane and Tatar [1] and others. The necessary conditions results are inspired by some arguments due to Baras and Kersner [17].

Now, we present two different definitions of fractional derivatives (see [13,18]).

We define the fractional derivative in the Caputo sense of power \( \mu \) by

\[
C^\mu D^\alpha u(t) := \frac{1}{\Gamma(n-\mu)} \int_0^t (t-\tau)^{n-\mu-1} u^{(n)}(\tau) d\tau, \quad n-1 < \mu < n.
\]

The fractional derivative in the Riemann-Liouville sense is given by

\[
R^\mu D^\alpha u(t) := \frac{1}{\Gamma(n-\mu)} \left( \frac{d}{dt} \right)^n \int_0^t (t-\tau)^{n-\mu-1} u(\tau) d\tau, \quad t > 0.
\]