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On a Class of Neumann Boundary Value Equations Driven by a $(p_1,\cdots,p_n)$-Laplacian Operator

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Abstract. In this paper we prove the existence of an open interval (λ',λ'') for each λ in the interval a class of Neumann boundary value equations involving the $(p_1,\ldots,p_n)$-Laplacian and depending on λ admits at least three solutions. Our main tool is a recent three critical points theorem of Averna and Bonanno [Topol. Methods Nonlinear Anal. [1] (2003) 93-103].

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1 Introduction

Here and in what follows, $\Omega\subset\mathbb{R}^N(N\geq 1)$ is a non-empty bounded open set with a boundary $\partial\Omega$ of class C^1, $p_i>N$ for $1\leq i\leq n$ and λ is a positive parameter.

Let us consider the following quasilinear elliptic system

$$\begin{cases} \Delta_{p_1}u_1+\lambda F_{u_1}(x,u_1,\cdots,u_n)=a_1(x)|u_1|^{p_1-2}u_1 & \text{in } \Omega,\\ \Delta_{p_2}u_2+\lambda F_{u_2}(x,u_1,\cdots,u_n)=a_2(x)|u_2|^{p_2-2}u_2 & \text{in } \Omega,\\ \vdots & \\ \Delta_{p_n}u_n+\lambda F_{u_n}(x,u_1,\cdots,u_n)=a_n(x)|u_n|^{p_n-2}u_n & \text{in } \Omega,\\ \dfrac{\partial u_i}{\partial \nu}=0 \quad \text{for } 1\leq i\leq n & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Delta_{p_i}u_i := \mathrm{div}(|\nabla u_i|^{p_i-2}\nabla u_i)$ is the p_i-Laplacian operator and ν is the outer unit normal to $\partial\Omega$. Here, $F:\Omega\times\mathbb{R}^n\to\mathbb{R}$ is a function such that the mapping $(t_1,t_2,\cdots,t_n)\to F(x,t_1,t_2,\cdots,t_n)$ is measurable in Ω for all $(t_1,\cdots,t_n)\in\mathbb{R}^n$ and is C^1 in $\mathbb{R}^n$ for almost every $x\in\Omega$ satisfying the condition

$$\sup_{\sum_{i=1}^n |t_i|^{p_i}/p_i\le\varrho} |F(\cdot,t_1,\cdots,t_n)|\in L^1(\Omega)$$

for every $\varrho>0$, F_{u_i} denotes the partial derivative of F with respect to u_i, and $a_i\in L^\infty(\Omega)$ with $\mathrm{essinf}_\Omega a_i\ge 0$ for $1\le i\le n$.

Throughout this paper, we let X be the Cartesian product of n spaces $W^{1,p_i}(\Omega)$ for $1\le i\le n$, i.e., $X=W^{1,p_1}(\Omega)\times W^{1,p_2}(\Omega)\times\cdots\times W^{1,p_n}(\Omega)$ equipped with the norm

$$\|(u_1,u_2,\cdots,u_n)\| := \|u_1\|+\|u_2\|+\cdots+\|u_n\|,$$

where

$$\|u_i\| := \Big(\int_\Omega |\nabla u_i(x)|^{p_i}\mathrm{d}x+\int_\Omega a_i(x)|u_i(x)|^{p_i}\mathrm{d}x\Big)^{\frac{1}{p_i}}$$

for $1\le i\le n$, which is equivalent to the usual one.

Put

$$c:=\max\Big\{\sup_{u_i\in W^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{\|u_i\|^{p_i}} : \text{ for } 1\le i\le n\Big\}. \tag{1.2}$$

Since $p_i>N$ for $1\le i\le n$, X is compactly embedded in $(C^0(\overline{\Omega}))^n$, so that $c<+\infty$. It follows from [2, Proposition 4.1] that

$$\sup_{u_i\in W^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|^{p_i}}{\|u_i\|^{p_i}}>\frac{1}{\|a_i\|_1} \qquad \text{for } 1\le i\le n,$$

where $\|a_i\|_1:=\int_\Omega |a_i(x)|\mathrm{d}x$ for $1\le i\le n$, and so $1/\|a_i\|_1\le c$ for $1\le i\le n$. In addition, if Ω is convex, it is known [2] that

$$\begin{aligned}&\sup_{u_i\in W^{1,p_i}(\Omega)\setminus\{0\}}\frac{\max_{x\in\overline{\Omega}}|u_i(x)|}{\|u_i\|}\\ &\le 2^{\frac{p_i-1}{p_i}}\max\Big\{\Big(\frac{1}{\|a_i\|_1}\Big)^{\frac{1}{p_i}},\ \frac{\mathrm{diam}(\Omega)}{N^{\frac{1}{p_i}}}\Big(\frac{p_i-1}{p_i-N}m(\Omega)\Big)^{\frac{p_i-1}{p_i}}\frac{\|a_i\|_\infty}{\|a_i\|_1}\Big\}\end{aligned}$$

for $1\le i\le n$, where $m(\Omega)$ is the Lebesgue measure of the set Ω, and equality occurs when Ω is a ball.

By a (weak) solution of the system (1.1), we mean any $u=(u_1,u_2,\cdots,u_n)\in X$ such that

$$\begin{aligned}&\int_\Omega\sum_{i=1}^n |\nabla u_i(x)|^{p_i-2}\nabla u_i(x)\nabla v_i(x)\mathrm{d}x\\ &\quad -\lambda\int_\Omega\sum_{i=1}^n F_{u_i}(x,u_1(x),\cdots,u_n(x))v_i(x)\mathrm{d}x+\int_\Omega\sum_{i=1}^n a_i(x)|u_i(x)|^{p_i-2}u_i(x)v_i(x)\mathrm{d}x=0\end{aligned}$$