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Inviscid Limit for Scalar Viscous Conservation Laws in Presence of Strong Shocks and Boundary Layers

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Abstract. In this paper, we study the inviscid limit problem for the scalar viscous conservation laws on half plane. We prove that if the solution of the corresponding inviscid equation on half plane is piecewise smooth with a single shock satisfying the entropy condition, then there exist solutions to the viscous conservation laws which converge to the inviscid solution away from the shock discontinuity and the boundary at a rate of ε^1 as the viscosity ε tends to zero.

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1 Introduction

In the theory of compressible fluids, the basic physics issue motivating the mathematical problem is the asymptotic equivalence between the viscous flows and the associated inviscid flows in the limit of small viscosity. This problem is particularly important and of great significance in many physical phenomena and their numerical computations in the presence of boundaries and shock discontinuities. When the underlying inviscid flow is smooth, the Cauchy problem can be solved by classical methods. However, in the presence of boundaries or shock discontinuities, the viscous flows display turbulent behavior in the limit of small viscosity, see the studies [1–8] and the references therein. The rigorous mathematical justification of this asymptotic equivalence poses challenging problems in many important cases. Here we consider the case in presence of both boundary and shock discontinuity for the scalar conservation laws. The goals of the present paper are

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to understand the evolution and structure of viscous boundary layers and shock layers and their interactions with interior inviscid hyperbolic flows and to show the uniform convergence of the viscous solutions to the piecewise smooth inviscid flow away from the boundary and shock discontinuity.

We consider the boundary value problem of the scalar viscous conservation laws

$$\begin{cases} u_t^\varepsilon + f(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon, & x>0,\ t>0, \\ u^\varepsilon(0,t) = u_1(t), & t>0, \end{cases} \tag{1.1}$$

where u_1 and $f=f(\cdot)$ are smooth known functions, and $\varepsilon>0$ is the viscosity constant. We require that

$$f'(u)<0 \qquad \text{and} \qquad f''(u)>0, \tag{1.2}$$

in the region in which we are interested. The corresponding inviscid hyperbolic equation is expressed as

$$u_t + f(u)_x = 0. \tag{1.3}$$

We impose (1.3) with the following discontinuous initial data

$$u(x,0) = u_0(x) \equiv \begin{cases} u_-(x), & x<x_0, \\ u_+(x), & x>x_0, \end{cases} \tag{1.4}$$

for some large $x_0>0$, where $u_\pm$ are smooth functions satisfying

$$\lim_{x\to x_0^+} u_+(x) < \lim_{x\to x_0^-} u_-(x). \tag{1.5}$$

By the condition (1.2), we need not to impose boundary condition to the inviscid equation (1.3). We consider the case that there is a shock solution u to the problem (1.3) with the shock issuing from the point $x_0>0$.

Definition 1.1. *A function $u(x,t)$ is called a shock solution of* (1.3) *up to time T if*

i) *$u(x,t)$ is a distributional solution of the hyperbolic equation* (1.3) *in the region* $(0,\infty)\times[0,T]$.

ii) *There is a smooth curve, the shock, $x=s(t)$, $s(0)=x_0$, $0\le t\le T$, so that $u(x,t)$ is sufficiently smooth at any point $x\neq s(t)$.*

iii) *The limits*

$$\partial_x^l u(s(t)-0,t) = \lim_{x\to s(t)^-} \partial_x^l u(x,t),$$
$$\partial_x^l u(s(t)+0,t) = \lim_{x\to s(t)^+} \partial_x^l u(x,t),$$

exist and are finite for $t\le T$ and $0\le l\le 5$.

iv) *The Lax geometrical entropy condition* [9] *is satisfied at $x=s(t)$, that is,*

$$f'(u(s(t)+0,t)) < \dot{s} < f'(u(s(t)-0,t)), \tag{1.6}$$

where $\dot{s}=\frac{d}{dt}s(t)$. This condition implies that $\dot{s}<0$.