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The Blow-Up Phenomena for the Camassa-Holm Equation with a Zero Order Dissipation

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Abstract. In this paper, we study the Cauchy problem of the Camassa-Holm equation with a zero order dissipation. We establish local well-posedness and investigate the blow-up phenomena for the equation.

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1 Introduction

The Camassa-Holm (CH) equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \qquad t > 0, \quad x \in \mathbb{R},$$

is a model for wave motion on shallow water, where $u(t,x)$ represents the fluid's free surface above a flat bottom (or equivalently, the fluid velocity at time $t \geq 0$ in the spatial x direction). The CH equation was first derived by Fokas and Fuchassteiner [1] as a bi-Hamiltonian system, and then as a model for shallow water waves by Camassa and Holm [2, 3].

Camassa and Holm [2] found that the solitary waves of CH equation are

$$u_c(x,t) = ce^{-|x-ct|}, \qquad x \in \mathbb{R}.$$

Moreover, the solitary waves are solitons: they retain their individuality under interaction and eventually emerge with their original shapes and speeds (see [2–4]). The solitons

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are peaked waves [5,6] and have to be understood [19] as weak solutions of this equation. Its solitary waves are orbitally stable [4, 6, 7] and analogous to the exact traveling wave solutions of the governing equations for water waves representing waves of great height – see the recent discussions in [8,9]. Other interesting aspects of CH equation are its complete integrability [3, 5, 10] and the fact that it is a re-expression of geodesic flow on the diffeomorphism group of the circle [11] and geodesic exponential maps of the Virasoro group [12].

The Cauchy problem of CH equation has been studied extensively. It has been shown that the equation is locally well-posed [13–18] for initial data $u_0 \in H^s(\mathbb{S})$ with $s > 3/2$. More interestingly, it has not only global strong solutions modelling permanent waves [13, 17–20] but also blow-up solutions modelling wave breaking [13–20]. The fact that the solitons do not belong to the space $H^s(\mathbb{R})$ with $s > 3/2$ motivates the study of weak solutions to the problem that are suitable to treat soliton interaction (see [21,22,29]). The advantage of the CH equation in comparison with the KdV equation lies in the fact that the CH equation has peaked solitons and models wave breaking [3, 13, 14].

It is difficult to avoid energy dissipation mechanisms in a real world. Ott and Sudan [23] investigated how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of the KdV equation, and Ghidaglia [24] investigated the long time behavior of solutions to the weakly dissipative KdV equation as a finite dimensional dynamical system.

We would like to consider the dissipative Camassa-Holm equation:

$$u_t - u_{txx} + 3uu_x + L(u) = 2u_x u_{xx} + uu_{xxx}, \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $L(u)$ is a dissipative term, L can be a differential operator or a quasi-differential operator according to different physical situations. We are interested in the effect of the weakly dissipative term on the CH equation.

The local well-posedness, global existence and blow-up phenomena of the Cauchy problem of the weakly dissipative CH equation (1.1) with $L = 1 - \partial_x^2$ were studied recently [25, 26]. We found that the behaviors of the weakly dissipative CH equation are similar to the CH equation in a finite interval of time, such as, the local well-posedness and the blow-up phenomena, and that there are considerable differences between the weakly dissipative CH equation and the CH equation in their long time behaviors. The global solutions of the weakly dissipative CH equation decay to zero as time goes to infinite provided the initial potential $y_0 = (1 - \partial_x^2)u_0$ is of some sign conditions (see [25,26]). This long time behavior is an important feature that the CH equation does not possess. In addition, the weakly dissipative CH equation has not the following conservation laws:

$$I_1 = \int_{\mathbb{R}} u \mathrm{d}x, \qquad I_2 = \int_{\mathbb{R}} (u^2 + u_x^2) \mathrm{d}x,$$

which play an important role in the study of the CH equation.

In this paper, we will study the Camassa-Holm equation with a zero order dissipation:

$$u_t - u_{txx} + 3uu_x + \lambda u = 2u_x u_{xx} + uu_{xxx}, \qquad t > 0, \quad x \in \mathbb{R}, \tag{1.2}$$