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# Local Well-Posedness and Blow Up Criterion for the Inviscid Boussinesq System in Hölder Spaces

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**Abstract.** We prove the local in time existence and a blow up criterion of solution in the Hölder spaces for the inviscid Boussinesq system in $\mathbb{R}^N, N\geq 2$, under the assumptions that the initial values $\theta_0, u_0\in C^r$, with $1<r\neq 2$.

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**Key Words**: Inviscid Boussinesq system; local well-posedness; blow-up criterion.

## 1 Introduction

The Cauchy problem for the Boussinesq system in $\mathbb{R}^N (N\geq 2)$ can be written as

$$\begin{cases} \partial_t\theta+u\cdot\nabla\theta-\kappa\Delta\theta=g, \\ \partial_t u+u\cdot\nabla u+\nabla\Pi-\nu\Delta u=f, \\ \mathrm{div} u=0, \end{cases} \tag{1.1}$$

with initial data

$$\theta|_{t=0}=\theta_0, \qquad u|_{t=0}=u_0. \tag{1.2}$$

Here $u(x,t),(x,t)\in\mathbb{R}^N\times(0,\infty), N\geq 2$, is the velocity vector field, $\theta(x,t)$ is the scalar temperature, $\Pi(x,t)$ is the scalar pressure, $f(x,t)$ is the external forces, which is a vector

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function, and $g$ is a known scalar function. $\nu \geq 0$ is the kinematic viscosity, and $\kappa \geq 0$ is the thermal diffusivity.

The Boussinesq system is extensively used in the atmospheric sciences and oceanographic turbulence in which rotation and stratification are important (see [1,2] and references therein). When $\kappa>0$ and $\nu>0$, the 2D Boussinesq system with $g=0, f=\theta e_2 (e_2=(0,1))$ has been well-understood (see [3–5]). When $\kappa>0$ and $\nu=0$ or $\kappa=0$ and $\nu>0$, the Boussinesq system is usually called the partial viscosity one. In these cases, the Boussinesq system has also been extensively and successfully studied. In particular, in the case $\kappa>0$ and $\nu=0$, Chae [6] proved that the Boussinesq system is globally well-posedness in $\mathbb{R}^m$ for any $m \geq 3$ and this result was extended by Hmidi and Keraani [7], Danchin and Paicu [8] to rough initial data in Besov space framework. In the case $\kappa=0$ and $\nu>0$, the global well-posedness was proved by Chae [6], Hou and Li [9] in $H^m(\mathbb{R}^2))$ space with $m \geq 3$.

When $\kappa=0$ and $\nu=0$, the Boussinesq system (1.1) becomes the inviscid one. In this case, it is clear that if $\theta \equiv 0$, the inviscid Boussinesq system reduces to the classical Euler equations. And the two-dimensional Boussinesq system can be used as a model for the three-dimensional axisymmetric Euler equations with swirl (see [10]). However, the global well-posedness problem of the inviscid Boussinesq system is still completely open in general (an exceptional case is the two-dimensional Euler equations which correspond to $\theta \equiv 0$, see [11] and references therein). Local existence and blow-up criteria have been established for the inviscid Boussinesq system (see [12–15] and references therein). In particular, Chae et al. considered in [12] the inviscid Boussinesq system with $g=0$ and $f=\theta f_1$, where $f_1$ satisfies that $curl f=0$ and $f \in L^\infty_{loc}([0,\infty);W^{1,\infty}(\mathbb{R}^2))$. They proved that there exists a unique and local $C^{1+\gamma}$ solution of the inviscid Boussinesq system with initial data $u_0,\theta_0 \in C^{1+\gamma}, \omega_0, \Delta\theta_0 \in L^q$ for $0<\gamma<1$ and $1<q<2$, where $\omega_0$ is the initial vorticity of the initial velocity $u_0$. They also proved a blow-up criterion for the local solution, which says that the gradient of the passive scalar $\theta$ controls the breakdown of $C^{1+\gamma}$ solutions of the Boussinesq system.

In this paper, we devote to the following inviscid Boussinesq system

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ \partial_t u + u \cdot \nabla u + \nabla \Pi = \theta e_N, \\ \mathrm{div} u = 0, \end{cases} \tag{1.3}$$

with the initial data

$$\theta|_{t=0} = \theta_0, \qquad u|_{t=0} = u_0. \tag{1.4}$$

In [11], the local in time existence and blow up criterion for the Euler equations with the initial data $u_0 \in C^r (r>1)$ was proved. Here we will extend the approaches and results in [11] to the inviscid Boussinesq system (1.3)-(1.4) with $r>1$ and $r \neq 2$. The case $r \neq 2$ is an extra case since we can not obtain the convergence of the approximate solutions up to now (see Section 4.2). We are confident that in this case our results keep true and we will work on it later. It should be noted that our results will relax the initial conditions in [12].