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The Integrability of Dispersive Hunter-Saxton Equation

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Abstract. In this paper, we prove that the dispersive form of Hunter-Saxton equation is a completely integrable and bi-Hamiltonian system.

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1 Introduction

In this paper we investigate the integrability of the following dispersive Hunter-Saxton equation

$$(u_t+uu_x+u_{xxx})_x=\frac{1}{2}u_x^2. \tag{1.1}$$

More concretely, we construct a Lax pair and bi-Hamiltonian functionals with corresponding conservation laws for (1.1). Eq. (1.1) can be regarded as a dispersively perturbed equation of the following Hunter-Saxton equation

$$(u_t+uu_x)_x=\frac{1}{2}u_x^2. \tag{1.2}$$

Eq. (1.2) can be derived as the leading order equation of a variational wave equation, which describes the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal director field [1], and it can also be derived as the high-frequency limit of the Camassa-Holm equation. The equation has a unique global weak solution which conserves energy even after their derivative blows up (see [2] for instance). In [3],

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Hunter and Zheng found its bi-Hamiltonian structure and Lax representation, and the rich integrability remain globally valid for those global conservative weak solutions.

Our main aim in this work is to describe the integrability structure of (1.1), which will be illustrated in Theorem 2.1 and Theorem 3.1. As is pointed out in [1] (pp. 371-372), it is deserved to notice that there exists a weak solution of (1.1) when the initial data only belongs to $L^2(\mathbb{R})$. The energy of the weak solution is bounded by the initial energy, but it is not clear how to prove that the energy is constant in time. It is known for the Korteweg-de Vries equation that the integrability transforms rapid decay of initial data into the smoothness of the solution, which conserves the energy. Therefore, perhaps the integrability of (1.1), established in this paper, can be used in some way.

2 Lax representation

In this section we give a Lax representation. Here we denote the integral operator D^{-1} by

$$D^{-1}=\frac{1}{2}\left(\int_{-\infty}^{x}-\int_{x}^{+\infty}\right).$$

We consider a class of natural solutions u with u_x is compactly supported. Integrating (1.1) twice give the nonlocal evolution equation

$$u_t+uu_x+u_{xxx}=\frac{1}{2}D^{-1}(u_x^2). \tag{2.1}$$

Then we have

Theorem 2.1. *The Lax representation is as follows*

$$L_t+[L,M]=0,$$

where L and M are the operators

$$L=D^{-1}u_x-\frac{1}{4}D^{-2}u_{xx}-\frac{3}{2}D^2-u, \tag{2.1}$$

$$M=-uD-\frac{1}{2}D^{-1}u_{xx}-D^3+\frac{1}{2}u_x. \tag{2.2}$$