Blowing Up of Sign-Changing Solutions to an Elliptic Subcritical Equation

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Abstract. This paper is concerned with the following non linear elliptic problem involving nearly critical exponent $(P^k_\varepsilon)$: $(-\Delta)^k u = K(x)|u|^{(4k/(n-2k))-1}u$ in $\Omega$, $\Delta^{k-1}u = \cdots = \Delta u = u = 0$ on $\partial\Omega$, where $\Omega$ is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2k+2$, $k \geq 1$, $\varepsilon$ is a small positive parameter and $K$ is a smooth positive function in $\Omega$. We construct sign-changing solutions of $(P^k_\varepsilon)$ having two bubbles and blowing up either at two different critical points of $K$ with the same speed or at the same critical point.

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1 Introduction and results

In this paper, we consider the following semilinear equation under the Navier boundary condition:

$$
\begin{cases}
(-\Delta)^k u = K(x)|u|^{p-1-\varepsilon}u, & \text{in } \Omega, \\
\Delta^{k-1} u = \cdots = \Delta u = u = 0, & \text{on } \partial\Omega,
\end{cases}
(P^k_\varepsilon)
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2k+1$, $k \geq 1$, $\varepsilon$ is a positive real parameter, $p+1 = 2n/(n-2k)$ is the critical Sobolev exponent for the embedding of $H^k(\Omega)$ into $L^{p+1}(\Omega)$, and $K$ is a smooth positive function in $\Omega$.

The study of concentration phenomena of $(P^k_\varepsilon)$ has attracted considerable attention in the last decades. See for example [1–8] and the references therein. When $k \in \{1,2\}$, there are many works devoted to the study of the positive solutions of $(P^k_\varepsilon)$, see for examples [1,8–11]. In sharp contrast to this, very little study has been made concerning the sign-changing solutions. For example, for $K \equiv 1$, in the Laplacian case, Ben Ayed-El Mehdi-Pacella [4] studied the blow-up of low energy sign changing solutions and part of this

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result is generalized in the biharmonic operator case in [5]. The main difficulty in the biharmonic operator case compared with the Laplacian one is that $u^+ := \max(u, 0)$ and $u^- := \max(0, -u)$ do not belong to $H^2(\Omega)$. This difficulty persist for the problem $(P^k_\epsilon)$ for $k \geq 2$.

The purpose of the present paper is to construct solutions for $(P^k_\epsilon)$ concentrating at two different critical point of $K$ or at the same critical point. Our method uses some techniques introduced by A. Bahri [12] and developed by his students. The main idea consists in performing refined expansions of the Euler functional associated to our variational problem, and its gradient in a neighborhood of potential concentration sets. Such expansions are made possible through a finite dimension reduction argument.

We define the Hilbert space $\mathcal{H}(\Omega)$ by

$$\mathcal{H}(\Omega) := \left\{ u \in H^k(\Omega) \mid \Delta^m u = \cdots = \Delta u = 0 \right\},$$

with $m_k = m - 1$ if $k = 2m$ and $m_k = m$ if $k = 2m + 1$. $\mathcal{H}(\Omega)$ is equipped with the norm $\| \cdot \|$ and its corresponding inner product $(\cdot, \cdot)$ defined by

$$\| u \|^2 = \int_\Omega |Lu|^2; \quad (u, v) = \int_\Omega Lu L v, \quad u, v \in \mathcal{H}(\Omega),$$

where $L := \Delta^m$ if $k = 2m (m \in \mathbb{N}^*)$ and $L := \nabla (\Delta^m)$ if $k = 2m + 1 (m \in \mathbb{N})$.

Our problem has a variational structure. The related functional is

$$I_\epsilon(u) := \frac{1}{2} \| u \|^2 - \frac{1}{p + 1 - \epsilon} \int_\Omega K(y)|u|^{p + 1 - \epsilon}, \quad u \in \mathcal{H}(\Omega).$$

Note that each critical point of $I_\epsilon$ is a solution of $(P^k_\epsilon)$.

For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{(a, \lambda)}(x) = \frac{c_0 \lambda^{(n - 2k)/2}}{(1 + \lambda^2 |x - a|^2)^{(n - 2k)/2}},$$

where $c_0$ is a positive constant chosen so that $\delta_{(a, \lambda)}$ is the family of solutions of the following problem

$$(-\Delta)^k u = u^{(n + 2k)/(n - 2k)}, \quad u > 0, \quad \text{in } \mathbb{R}^n. \quad (1.3)$$

Notice that, the family $\delta_{(a, \lambda)}$ achieves the best Sobolev constant

$$S := \inf \left\{ \| Lu \|^2_{L^2(\mathbb{R}^n)} \| u \|^2_{L^{\frac{2n}{n-2k}}(\mathbb{R}^n)} : u \neq 0, Lu \in L^2(\mathbb{R}^n), \text{ and } u \in L^{\frac{2n}{n-2k}}(\mathbb{R}^n) \right\}.$$

Observe that, under Navier boundary conditions $(\Delta^{k-1} u = \cdots = \Delta u = u = 0$ on $\partial \Omega$), the operator $(-\Delta)^k$ satisfies the maximum principle. In the sequel we will denote by $G$ the Green’s function and by $H$ its regular part, that is,

$$G(x, y) = |x - y|^{2k - n} - H(x, y), \quad \text{for } (x, y) \in \Omega^2,$$