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Blowing Up of Sign-Changing Solutions to an Elliptic Subcritical Equation

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Abstract. This paper is concerned with the following non linear elliptic problem involving nearly critical exponent (P_ε^k): $(-\Delta)^k u = K(x)|u|^{(4k/(n-2k))-\varepsilon}u$ in Ω, $\Delta^{k-1}u = \cdots = \Delta u = u = 0$ on $\partial\Omega$, where Ω is a bounded smooth domain in $\mathbb{R}^n$, $n \geq 2k+2$, $k \geq 1$, ε is a small positive parameter and K is a smooth positive function in $\overline{\Omega}$. We construct sign-changing solutions of (P_ε^k) having two bubbles and blowing up either at two different critical points of K with the same speed or at the same critical point.

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1 Introduction and results

In this paper, we consider the following semilinear equation under the Navier boundary condition:

$$\begin{cases} (-\Delta)^k u = K(x)|u|^{p-1-\varepsilon}u, & \text{in } \Omega, \\ \Delta^{k-1}u = \cdots = \Delta u = u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_ε^k}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geq 2k+1$, $k \geq 1$, ε is a positive real parameter, $p+1 = 2n/(n-2k)$ is the critical Sobolev exponent for the embedding of $H^k(\Omega)$ into $L^{p+1}(\Omega)$, and K is a smooth positive function in $\overline{\Omega}$.

The study of concentration phenomena of (P_ε^k) has attracted considerable attention in the last decades. See for example [1–8] and the references therein. When $k \in \{1,2\}$, there are many works devoted to the study of the positive solutions of (P_ε^k), see for examples [1, 8–11]. In sharp contrast to this, very little study has been made concerning the sign-changing solutions. For example, for $K \equiv 1$, in the Laplacian case, Ben Ayed-El Mehdi-Pacella [4] studied the blow-up of low energy sign changing solutions and part of this

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result is generalized in the biharmonic operator case in [5]. The main difficulty in the biharmonic operator case compared with the Laplacian one is that $u_\varepsilon^+ := \max(u_\varepsilon, 0)$ and $u_\varepsilon^- := \max(0, -u_\varepsilon)$ do not belong to $H^2(\Omega)$. This difficulty persist for the problem (P_ε^k) for $k \geq 2$.

The purpose of the present paper is to construct solutions for (P_ε^k) concentrating at two different critical point of K or at the same critical point. Our method uses some techniques introduced by A. Bahri [12] and developed by his students. The main idea consists in performing refined expansions of the Euler functional associated to our variational problem, and its gradient in a neighborhood of potential concentration sets. Such expansions are made possible through a finite dimension reduction argument.

We define the Hilbert space $\mathcal{H}(\Omega)$ by

$$\mathcal{H}(\Omega) := \left\{ u \in H^k(\Omega) \,\middle|\, \Delta^{m_k} u = \cdots = \Delta u = u = 0 \right\},$$

with $m_k = m-1$ if $k = 2m$ and $m_k = m$ if $k = 2m+1$. $\mathcal{H}(\Omega)$ is equipped with the norm $\|\cdot\|$ and its corresponding inner product $(\cdot,\cdot)$ defined by

$$\|u\|^2 = \int_\Omega |Lu|^2; \qquad (u,v) = \int_\Omega Lu Lv, \quad u,v \in \mathcal{H}(\Omega),$$

where $L := \Delta^m$ if $k = 2m\,(m \in \mathbb{N}^*)$ and $L := \nabla(\Delta^m)$ if $k = 2m+1\,(m \in \mathbb{N})$.

Our problem has a variational structure. The related functional is

$$I_\varepsilon(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1-\varepsilon}\int_\Omega K(y)|u|^{p+1-\varepsilon}, \qquad u \in \mathcal{H}(\Omega). \tag{1.1}$$

Note that each critical point of I_ε is a solution of (P_ε^k).

For $a \in \Omega$ and $\lambda > 0$, let

$$\delta_{(a,\lambda)}(x) = \frac{c_0 \lambda^{(n-2k)/2}}{(1+\lambda^2|x-a|^2)^{(n-2k)/2}}, \tag{1.2}$$

where c_0 is a positive constant chosen so that $\delta_{(a,\lambda)}$ is the family of solutions of the following problem

$$(-\Delta)^k u = u^{(n+2k)/(n-2k)}, \qquad u > 0, \qquad \text{in } \mathbb{R}^n. \tag{1.3}$$

Notice that, the family $\delta_{(a,\lambda)}$ achieves the best Sobolev constant

$$S := \inf\left\{ \|Lu\|^2_{L^2(\mathbb{R}^n)} \|u\|^{-2}_{L^{\frac{2n}{n-2k}}(\mathbb{R}^n)} : u \not\equiv 0,\ Lu \in L^2(\mathbb{R}^n),\ \text{and } u \in L^{\frac{2n}{n-2k}}(\mathbb{R}^n) \right\}.$$

Observe that, under Navier boundary conditions ($\Delta^{k-1}u = \cdots = \Delta u = u = 0$ on $\partial\Omega$), the operator $(-\Delta)^k$ satisfies the maximum principle. In the sequel we will denote by G the Green's function and by H its regular part, that is,

$$G(x,y) = |x-y|^{2k-n} - H(x,y), \qquad \text{for } (x,y) \in \Omega^2,$$