JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **26**, No. 1, pp. 1-13

doi: 10.4208/jpde.v26.n1.1
March 2013

# Global Existence and Uniqueness of Solutions to Evolution $p$-Laplacian Systems with Nonlinear Sources

WEI Yingjie and GAO Wenjie*

*Institute of Mathematics, Jilin University, Changchun 130012, China.*

Received 25 February 2012; Accepted 1 December 2012

**Abstract.** This paper presents the global existence and uniqueness of the initial and boundary value problem to a system of evolution $p$-Laplacian equations coupled with general nonlinear terms. The authors use skills of inequality estimation and the method of regularization to construct a sequence of approximation solutions, hence obtain the global existence of solutions to a regularized system. Then the global existence of solutions to the system of evolution $p$-Laplacian equations is obtained with the application of a standard limiting process. The uniqueness of the solution is proven when the nonlinear terms are local Lipschitz continuous.

**AMS Subject Classifications**: 35A01, 35A02, 35G55

**Chinese Library Classifications**: O175.29, O175.4

**Key Words**: Global existence; uniqueness; degenerate; $p$-Laplacian systems.

## 1 Introduction

In this paper, we study the global existence and uniqueness of solutions to the initial and boundary value problem

$$u_{it}-\mathrm{div}(|\nabla u_i|^{p_i-2}\nabla u_i)=f_i(u_1,\cdots,u_m),\qquad (x,t)\in\Omega\times(0,T),\tag{1.1a}$$
$$u_i(x,0)=u_{i0}(x),\qquad x\in\Omega,\tag{1.1b}$$
$$u_i(x,t)=0,\qquad (x,t)\in\partial\Omega\times(0,T),\tag{1.1c}$$

where $p_i>2$, $i=1,2,\cdots,m$, $T>0$, $\Omega\subset R^n$ is an open connected bounded domain with smooth boundary $\partial\Omega$.

System (1.1a) models such as non-Newtonian fluids [1,2] and nonlinear filtration [3], etc. In the non-Newtonian fluids theory, $(p_1,p_2,\cdots,p_m)$ is a characteristic quantity of the

*Corresponding author. *Email addresses:* weiyj@jlu.edu.cn (Y. Wei), wjgao@jlu.edu.cn (W. Gao)

fluids. The fluids with $(p_1,p_2,\cdots,p_m)>(2,2,\cdots,2)$ are called dilatant fluids and those with $(p_1,p_2,\cdots,p_m)<(2,2,\cdots,2)$ are called pseudoplastics. If $(p_1,p_2,\cdots,p_m)=(2,2,\cdots,2)$, they are Newtonian fluids.

For $p_i=2$, $i=1,2$, many authors have studied the problem above; most of them studied global existence, uniqueness, boundedness, and blowup behavior of solutions, etc(see [4–10]). Some authors have derived sufficient conditions for the nonexistence of global solutions. Such conditions are usually related to the structure of $f_i$, $i=1,2$. And some authors have studied the uniqueness of the global solution and blow-up of the positive solution, with nonlinearities in the form of

$$f_1(u_1,u_2)=u_1^\alpha u_2^\beta, f_2(u_1,u_2)=u_1^\gamma u_2^\delta,$$

where $\alpha,\beta,\gamma,\delta$ are nonnegative numbers.

For $p_i>2$, $i=1,2$, in [11], the authors gave local existence and uniqueness theorem of solutions for the initial and boundary value problem on $\Omega\times(0,T_1)$, where $T_1\in(0,T)(T>0)$ could be very small.

It is our goal to prove results of global existence and uniqueness for the degenerate system of $m$ equations. Since the system is coupled with nonlinear terms, in general, the solutions of (1.1a)-(1.1c) will not exist for all time. Inspired by [12], in this paper, we study some special cases by stating constrains to nonlinear functions. The proof consists of two steps. First, we prove that the approximating problem admits a global solution; then we do some uniform estimates for these solutions. We mainly use skills of inequality estimation and the method of regularization to construct a sequence of approximation solutions, hence obtain existence of the solution to a regularized system of equations. By a standard limiting process, we obtain the existence of solutions to the system (1.1a)-(1.1c).

Systems (1.1a) degenerates when $\nabla u_i=0$. In general, there is no classical solution; therefore, we have to study the generalized solutions to the problem (1.1a)-(1.1c). The definition of generalized solutions is as follows:

**Definition 1.1.** *A nonnegative function $u=(u_1,\cdots,u_m)$ is called a generalized solution to the system* (1.1a)-(1.1c) *in $\Omega_T$, $T>0$, if $u_i\in L^\infty(\Omega_T)\cap L^{p_i}(0,T;W_0^{1,p_i}(\Omega))$, $u_{it}\in L^2(\Omega_T)$, satisfying*

$$\begin{aligned}&\int_\Omega u_i(x,T)\varphi_i(x,T)\mathrm{d}x+\iint_{\Omega_T}|\nabla u_i|^{p_i-2}\nabla u_i\nabla\varphi_i\mathrm{d}x\mathrm{d}t\\ =&\iint_{\Omega_T}(f_i(u)\varphi_i+\varphi_{it}u_i)\mathrm{d}x\mathrm{d}t+\int_\Omega u_{i0}(x)\varphi_i(x,0)\mathrm{d}x,\end{aligned}\tag{1.2}$$

*for any $\varphi_i\in C^1(\overline{\Omega}_T)$, s.t. $\varphi_i=0$, for $(x,t)\in\partial\Omega\times(0,T)$; and $u_i(x,t)=0,(x,t)\in\partial\Omega\times(0,T)$, where $i=1,2,\cdots,m$.*

## 2 Main results

In order to study the problem (1.1a)-(1.1c), we make the following assumptions: