JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **26**, No. 4, pp. 305-362

doi: 10.4208/jpde.v26.n4.3
December 2013

Refined Scattering and Hermitian Spectral Theory for Linear Higher-Order Schrödinger Equations

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Received 9 December 2012; Accepted 3 July 2013

Abstract. The Cauchy problem for a linear $2m$th-order Schrödinger equation

$$u_t = -\mathrm{i}(-\Delta)^m u, \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u|_{t=0} = u_0; \quad m \geq 1 \text{ is an integer,}$$

is studied, for initial data u_0 in the weighted space $L^2_{\rho^*}(\mathbb{R}^N)$, with $\rho^*(x) = \mathrm{e}^{|x|^\alpha}$ and $\alpha = \frac{2m}{2m-1} \in (1,2]$. The following **five** problems are studied:

(I) A sharp asymptotic behaviour of solutions as $t \to +\infty$ is governed by a discrete spectrum and a countable set Φ of the eigenfunctions of the linear rescaled operator

$$\mathbf{B} = -\mathrm{i}(-\Delta)^m + \frac{1}{2m} y \cdot \nabla + \frac{N}{2m} I, \quad \text{with the spectrum } \sigma(\mathbf{B}) = \Big\{ \lambda_\beta = -\frac{|\beta|}{2m}, |\beta| \geq 0 \Big\}.$$

(II) Finite-time blow-up local structures of nodal sets of solutions as $t \to 0^-$ and a formation of "multiple zeros" are described by the eigenfunctions, being *generalized Hermite polynomials*, of the "adjoint" operator

$$\mathbf{B}^* = -\mathrm{i}(-\Delta)^m - \frac{1}{2m} y \cdot \nabla, \quad \text{with the same spectrum } \sigma(\mathbf{B}^*) = \sigma(\mathbf{B}).$$

Applications of these spectral results also include: **(III)** a unique continuation theorem, and **(IV)** boundary characteristic point regularity issues.

Some applications are discussed for more general linear PDEs and for the nonlinear Schrödinger equations in the focusing ("$+$") and defocusing ("$-$") cases

$$u_t = -\mathrm{i}(-\Delta)^m u \pm \mathrm{i}|u|^{p-1} u, \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } p > 1,$$

as well as for: **(V)** the quasilinear Schrödinger equation of a "porous medium type"

$$u_t = -\mathrm{i}(-\Delta)^m (|u|^n u), \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad \text{where } n > 0.$$

For the latter one, the main idea towards countable families of *nonlinear eigenfunctions* is to perform a homotopic path $n \to 0^+$ and to use spectral theory of the pair $\{\mathbf{B}, \mathbf{B}^*\}$.

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AMS Subject Classifications: 35K55, 35K40

Chinese Library Classifications: O175.26, O175.29

Key words: Higher-order Schrödinger operators; rescaled blow-up variables; discrete real spectrum; asymptotic behavior; nodal sets of solutions; unique continuation; boundary characteristic point regularity; quasilinear Schrödinger equations; nonlinear eigenfunctions.

1 Introduction: duality of global and blow-up scalings, Hermitian spectral theory, and refined scattering

1.1 Basic Shrödinger equations and key references

Consider the linear $2m$th-order Schrödinger equation (the LSE–$2m$), with any integer $m \geq 1$,

$$u_t = -\mathrm{i}(-\Delta)^m u, \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \quad u|_{t=0} = u_0, \tag{1.1}$$

where Δ is the Laplace operator in $\mathbb{R}^N$, for initial data u_0 in some weighted L^2-space, to be introduced. Here $m=1$ corresponds to the classic Schrödinger equation

$$\mathrm{i}u_t = -\Delta u, \quad \text{in } \mathbb{R}^N \times \mathbb{R}_+, \tag{1.2}$$

which very actively entered general PDE theory from Quantum Mechanics since 1920s.

It is not an exaggeration to say that, nowadays, linear and nonlinear Schrödinger type equations are the most popular PDE models of modern mathematics among other types of equations. In Appendix A, we present a "mathematical evidence" for that by using simple data from the `MathSciNet`. It not possible to express how deep is mathematical theory developed for models such as (1.2), (1.1), and related semilinear ones. We refer to well-known monographs [1,2], which cover classes of both linear and nonlinear PDEs.

Concerning the results that are more closely related to the subject of this paper, we note that scattering L^2- and $L^{q,r}_{x,t}$- theories for (1.2) have been fully developed in the works by Stein, Tomas, Segal, Strichartz in the 1970s, with later further involved estimates in more general spaces by Ginibre and Velo, Yajima, Cazenave and Weissler, Montgomery-Smith, Keel, Tao, and many others; see [3] and [4] for references concerning these, as well as optimal $L^{q,r}_{x,t}$-estimates for the non-homogeneous equation

$$\mathrm{i}u_t = -\Delta u + F(x,t), \quad \text{in } \mathbb{R}^N \times \mathbb{R}, \tag{1.3}$$

as well as more recent papers [5–7].

The $2m$th-order counterpart (1.1) was also under scrutiny for a long period. We refer to Ablowitz–Segur's monograph [8], Ivano–Kosevich [9], Turitsyn [10], Karpman [11], and Karpman–Shagalov [12] for physical, symmetry, and other backgrounds of higher-order Schrödinger-type semilinear models (see also [13] for extra motivations from nonlinear optics), [14] for first existence and uniqueness results, and more recent papers [15–24] as an account for further applied and rigorous research, as well as other earlier key references and surveys in this fundamental area of modern PDE theory.