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L^1 Existence and Uniqueness of Entropy Solutions to Nonlinear Multivalued Elliptic Equations with Homogeneous Neumann Boundary Condition and Variable Exponent

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Abstract. In this work, we study the following nonlinear homogeneous Neumann boundary value problem $\beta(u)-\mathrm{div}a(x,\nabla u)\ni f$ in Ω, $a(x,\nabla u)\cdot\eta=0$ on $\partial\Omega$, where Ω is a smooth bounded open domain in $\mathbb{R}^N$, $N\geq 3$ with smooth boundary $\partial\Omega$ and η the outer unit normal vector on $\partial\Omega$. We prove the existence and uniqueness of an entropy solution for L^1-data f. The functional setting involves Lebesgue and Sobolev spaces with variable exponent.

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1 Introduction

The paper is motivated by phenomena which are described by the homogeneous Neumann boundary value problem of the form

$$\begin{cases} \beta(u)-\mathrm{div}a(x,\nabla u)\ni f, & \text{in } \Omega, \\ a(x,\nabla u)\cdot\eta=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where η is the unit outward normal vector on $\partial\Omega$, Ω is a smooth bounded open domain in $\mathbb{R}^N, N\geq 3, \beta=\partial j$ is a maximal monotone graph in $\mathbb{R}^2$ with $\mathrm{dom}(\beta)$ bounded on $\mathbb{R}$ and $0\in\beta(0)$, $f\in L^1(\Omega)$ and a is a Leray-Lions operator which involves variable exponents.

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Note that j is a nonnegative, convex and l.s.c. function on $\mathbb{R}$ and, ∂j is the subdifferential of j. We set

$$\overline{\mathrm{dom}(\beta)}=[m,M]\subset\mathbb{R} \text{ with } m\leq 0\leq M.$$

Recall that a Leray-Lions operator which involves variable exponents is a Carathéodory function $a(x,\xi):\Omega\times\mathbb{R}^N\longrightarrow\mathbb{R}^N$ (i.e. $a(x,\xi)$ is continuous in ξ for a.e. $x\in\Omega$ and measurable in x for every $\xi\in\mathbb{R}^N$) such that:

• There exists a positive constant C_1 such that

$$|a(x,\xi)|\leq C_1(j(x)+|\xi|^{p(x)-1}), \tag{1.2}$$

for almost every $x\in\Omega$ and for every $\xi\in\mathbb{R}^N$ where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$, with $1/p(x)+1/p'(x)=1$.

• The following inequalities hold

$$(a(x,\xi)-a(x,\eta))\cdot(\xi-\eta)>0, \tag{1.3}$$

for almost every $x\in\Omega$ and for every $\xi,\eta\in\mathbb{R}^N$, with $\xi\neq\eta$, and

$$\frac{1}{C}|\xi|^{p(x)}\leq a(x,\xi)\cdot\xi, \tag{1.4}$$

for almost every $x\in\Omega, C>0$ and for every $\xi\in\mathbb{R}^N$.

In this paper, we make the following assumption on the variable exponent:

$$p(\cdot):\overline{\Omega}\to\mathbb{R} \text{ is a continuous function such that } 1<p_-\leq p_+<+\infty, \tag{1.5}$$

where $p_-:=\mathrm{essinf}_{x\in\Omega}\,p(x)$ and $p_+:=\mathrm{esssup}_{x\in\Omega}\,p(x)$.

As the exponent $p(\cdot)$ appearing in (1.2) and (1.4) depends on the variable x, the functional setting for the study of problem (1.1) involves Lebesgue and Sobolev spaces with variable exponents $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$. In the next section, we will make a brief presentation of the variable exponent spaces.

Many results are known as regards to elliptic problems in the variational setting for Dirichlet or Dirichlet-Neumann problems (cf. [1–9]).

Problem (1.1) can be viewed as an extension of the following

$$\begin{cases} b(u)-\mathrm{div}a(x,\nabla u)=f, & \text{in } \Omega,\\ a(x,\nabla u)\cdot\eta=0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

where Ω is a smooth bounded open domain in $\mathbb{R}^N, N\geq 3$ and η the outer unit normal vector on $\partial\Omega$. $b:\mathbb{R}\to\mathbb{R}$ is a continuous, nondecreasing function, surjective such that $b(0)=0$, $f\in L^1(\Omega)$ and a is a Lerray-Lions operator which involves variable exponents.

Problem (1.6) was studied by Bonzi, Nyanquini and Ouaro (cf. [2]) where they proved the existence and uniqueness of an entropy solution. An equivalent notion of solution is