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Existence of Renormalized Solutions for Nonlinear Parabolic Equations

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Abstract. We give an existence result of a renormalized solution for a class of nonlinear parabolic equations

$$\frac{\partial b(x,u)}{\partial t}-\operatorname{div}\Big(a(x,t,u,\nabla u)\Big)+g(x,t,u,\nabla u)+H(x,t,\nabla u)=f, \quad \text{in } Q_T,$$

where the right side belongs to $L^{p'}(0,T;W^{-1,p'}(\Omega))$ and where $b(x,u)$ is unbounded function of u and where $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions type operator with growth $|\nabla u|^{p-1}$ in ∇u. The critical growth condition on g is with respect to ∇u and no growth condition with respect to u, while the function $H(x,t,\nabla u)$ grows as $|\nabla u|^{p-1}$.

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1 Introduction

In the present paper, we study a nonlinear parabolic problem of the type

$$\begin{cases} \dfrac{\partial b(x,u)}{\partial t}-\operatorname{div}\Big(a(x,t,u,\nabla u)\Big)+g(x,t,u,\nabla u)+H(x,t,\nabla u)=f, & \text{in } Q_T,\\ b(x,u)(t=0)=0, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega\times(0,T), \end{cases} \tag{1.1}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$, $N\geq 1$, $T>0$, $p>1$ and Q_T is the cylinder $\Omega\times(0,T)$. The operator $-\operatorname{div}\big(a(x,t,u,\nabla u)\big)$ is a Leray-Lions operator which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u, the function $b(x,u)$ is an unbounded on u. The functions g and H are two the Carathéodory functions with suitable assumptions (see Assumption (H_2)). Finally the data f is in $L^{p'}(0,T;W^{-1,p'}(\Omega))$. We are interested in proving an existence result to (1.1). The difficulties connected to this problem are due to the data and the presence of the two terms g and H which induce a lack of coercivity.

For $b(x,u)=u$, the existence of a weak solution to Problem (1.1) (which belongs to $L^m(0,T;W_0^{1,m}(\Omega))$ with $p>2-1/(N+1)$ and $m<(p(N+1)-N)/N+1$ was proved in [1] (see also [2]) when $g=H=0$, and in [3] when $g=0$, and in [4–6] when $H=0$. In the present paper we prove the existence of renormalized solutions for a class of nonlinear parabolic problems (1.1). The notion of renormalized solution was introduced by Diperna and Lions [7] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1.1) by Boccardo et al. [8] when the right hand side is in $W^{-1,p'}(\Omega)$, by Rakotoson [9] when the right hand side is in $L^1(\Omega)$, and finally by Dal Maso, Murat, Orsina and Prignet [10] for the case of right hand side is general measure data.

In the case where $H=0$ and where the function $g(x,t,u,\nabla u)\equiv g(u)$ is independent on the $(x,t,\nabla u)$ and g is continuous, the existence of a renormalized solution to Problem (1.1) is proved in [11]. The case $H=0$ is studied by Akdim et al. (see [12,13]). The case $H=0$ and where g depends on (x,t,u) is investigated in [14]. In [15] the authors prove the existence of a renormalized solution for the complete operator. The case $g(x,t,u,\nabla u)\equiv\operatorname{div}(\phi(u))$ and $H=0$ is studied by Redwane in the classical Sobolev spaces $W^{1,p}(\Omega)$ and Orlicz spaces see [16,17], and where $b(x,u)=u$ (see [18]).

The aim of the present paper we prove an existence result for renormalized solutions to a class of problems (1.1) with the two lower order terms. It is worth noting that for the analogous elliptic equation with two lower order terms (see e.g. [19,20]). The plan of the article is as follows. In Section 2 we make precise all the assumptions on b, a, g, H, f and give the definition of a renormalized solution of (1.1). In Section 3 we establish the existence of such a solution (Theorem 3.1).

2 Basic assumptions on the data and definition of a renormalized solution

Throughout the paper, we assume that the following assumptions hold true:

Assumption (H1)

Let Ω be a bounded open set of $\mathbb{R}^N$ $(N\geq 1)$, $T>0$ is given and we set $Q_T=\Omega\times(0,T)$, and

$$b:\Omega\times\mathbb{R}\to\mathbb{R} \text{ is a Carathéodory function,} \tag{2.1}$$

such that for every $x\in\Omega$, $b(x,.)$ is a strictly increasing C^1-function with $b(x,0)=0$.