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Multiple Positive Solutions for Semilinear Elliptic Equations Involving Subcritical Nonlinearities in $\mathbb{R}^{\mathbb{N}}$

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Abstract. In this paper, we study how the shape of the graph of $a(z)$ affects on the number of positive solutions of

$$-\Delta v+\mu b(z)v=a(z)v^{p-1}+\lambda h(z)v^{q-1}, \qquad \text{in } \mathbb{R}^N. \tag{0.1}$$

We prove for large enough $\lambda,\mu>0$, there exist at least $k+1$ positive solutions of the this semilinear elliptic equations where $1\leq q<2<p<2^*=2N/(N-2)$ for$N\geq 3$.

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1 Introduction

For $N\geq 3$, $1\leqslant q<2<p<2^*=2N/(N-2)$, we suppose the semilinear elliptic equations

$$\begin{cases} -\Delta v+\mu b(z)v=a(z)v^{p-1}+\lambda h(z)v^{q-1}, & \text{in } \mathbb{R}^N;\\ v\in H^1(\mathbb{R}^N), \end{cases} \tag{$E_{\lambda,\mu}$}$$

where $\lambda,\mu>0$. Suppose a, b and h satisfy the following conditions:
(a_1) a is a positive continuous function in $\mathbb{R}^N$ and $\lim_{|z|\to\infty}a(z)=a_\infty>0$.
(a_2) There are k points $a^1,a^2,\cdots,a^k$ in $\mathbb{R}^N$ such that $a(a^i)=a_{\max}=\max_{z\in\mathbb{R}^N}a(z)$; for $1\leq i\leq k$ and $a_\infty<a_{\max}$.

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(h_1) $h\in L^{\frac{p}{p-q}}(\mathbb{R}^{\mathbb{N}})\cap L^{\infty}(\mathbb{R}^{\mathbb{N}})$ and $h\gneqq 0$.
(b_1) b is a bounded and positive continuous function in $\mathbb{R}^N$.

For $\mu=1$, $\lambda=0$, $a(z)=b(z)=1$ for all $z\in\mathbb{R}^N$, we assume the semilinear elliptic equation

$$\begin{cases} -\Delta u+u=u^{p-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^N), \end{cases} \tag{E_0}$$

where

$$\|u\|_H^2=\int_{\mathbb{R}^{\mathbb{N}}}(|\nabla u|^2+u^2)\mathrm{d}z \quad \text{is the norm in } H^1(\mathbb{R}^N),$$

and the energy functional

$$J_0^{\infty}(u)=\frac{1}{2}\|u\|_H^2-\frac{1}{p}\|u_+\|_{L^p}^p, \qquad \text{where } \ u_+=\max\{u,0\}\geqslant 0.$$

We consider the semilinear elliptic equation

$$\begin{cases} -\Delta u+u=a(z)u^{p-1}+\lambda h(z)u^{q-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^{\mathbb{N}}), \end{cases}$$

have been studied by Huei-li Lin [1] ($b(z)=1$, $\mu=1$ and for $N\geq 3$, $1\leqslant q<2<p<2^*=2N/(N-2)$) and she studied the effect of the coefficient a(z) of the subcritical nonlinearity in $\mathbb{R}^{\mathbb{N}}$, Ambrosetti [2] ($a\equiv 1$ and $1<q<2<p\leq 2^*=2N/(N-2)$ and Wu [3] ($a\in C(\overline{\Omega})$ and changes sign, $1<q<2<p<2^*$). They showed that this equation has at least two positive solutions for small enough $\lambda>0$. In [4], Hsu and Lin have studied that there are four positive solutions of the general cases

$$-\Delta v+v=a(z)v^{p-1}+\lambda h(z)v^{q-1}, \qquad \text{in } \mathbb{R}^N;$$

for small enough $\lambda>0$.

In this paper, we study the existence and multiplicity of positive solutions of the equation $(E_{\lambda,\mu})$ in $\mathbb{R}^{\mathbb{N}}$. By the change of variables

$$\mu=\frac{1}{\varepsilon^2} \qquad \text{and} \qquad u(z)=\varepsilon^{\frac{2}{p-2}}v(\varepsilon z),$$

Eq. $(E_{\lambda,\mu})$ is converted to

$$\begin{cases} -\Delta u+b(\varepsilon z)u=a(\varepsilon z)u^{p-1}+\lambda h(\varepsilon z)u^{q-1}, & \text{in } \mathbb{R}^{\mathbb{N}};\\ u\in H^1(\mathbb{R}^N). \end{cases} \tag{$E_{\varepsilon,\lambda}$}$$

Based on Eq. $(E_{\varepsilon,\lambda})$, we consider the C^1-functional $J_{\varepsilon,\lambda}$, for $u\in H^1(\mathbb{R}^{\mathbb{N}})$.

$$J_{\varepsilon,\lambda}(u)=\frac{1}{2}\int_{\mathbb{R}^{\mathbb{N}}}(|\nabla u|^2+b(\varepsilon z)u^2)\mathrm{d}z-\frac{1}{p}\int_{\mathbb{R}^{\mathbb{N}}}a(\varepsilon z)u_+^p\mathrm{d}z-\frac{1}{q}\int_{\mathbb{R}^{\mathbb{N}}}\lambda h(\varepsilon z)u_+^q\mathrm{d}z,$$