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Jacobi Elliptic Numerical Solutions for the Time Fractional Variant Boussinesq Equations

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Abstract. The fractional derivatives in the sense of Caputo, and the homotopy perturbation method are used to construct the approximate solutions for nonlinear variant Boussinesq equations with respect to time fractional derivative. This method is efficient and powerful in solving wide classes of nonlinear evolution fractional order equations.

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1 Introduction

Fractional differential equations have caught much attention recently due to the exact description of nonlinear phenomena. First there were almost no practical applications of fractional calculus, and it was considered by many authors as an abstract area containing only mathematical manipulations of little or no use. Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade Fractional Calculus has been applied to almost every field of science, engineering, and mathematics. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Many applications of fractional calculus can be found in turbulence and fluid dynamics, stochastic dynamic system, plasma physics and controlled thermonuclear fusion, nonlinear control theory, image processing, nonlinear biological systems, astrophysics [1–11]. Historical summaries of the developments of fractional calculus can

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be found in [1–3]. There has been some attempt to solve linear problems with multiple fractional derivatives (the so-called multi-term equations) [2, 12]. Not much work has been done for nonlinear problems and only a few numerical schemes have been proposed to solve nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivative and this motivates us to develop a numerical scheme for their solutions [13]. Most of fractional differential equations do not have exact analytical solutions, hence considerable heed has been focused on the approximate and numerical solutions such as Adomian decomposition method [14–17], variational iteration method [18, 19], homotopy perturbation method [20–22], homotopy Analysis method [23,24] and so on.

Consider the nonlinear variant Boussinesq equations [25]

$$\begin{cases} h_t+(hu)_x+u_{xxx}=0, \\ u_t+h_x+uu_x=0. \end{cases} \tag{1.1}$$

Lu [25] used the Jacobi elliptic functions to obtain the exact solutions for two variant Boussinesq equations (1.1). Zayed et al. [26] have used the homotopy perturbation method and Adomian decomposition method to introduce the approximate solutions for nonlinear variant Boussinesq equations (1.1).

In this article, we give a new model of the time fraction variant Boussinesq equations of the form

$$\begin{cases} D_t^\alpha h+(hu)_x+u_{xxx}=0, \\ D_t^\alpha u+h_x+uu_x=0, \quad 0<\alpha\leq 1, \end{cases} \tag{1.2}$$

where $D_t^\alpha=\partial^\alpha/\partial t^\alpha$.

This system has been discussed by many authors when $\alpha\to 1$. We use homotopy perturbation method to calculate an approximate solution of time fraction variant Boussinesq equations (1.2) which is a generalization of the given solution in [25,26].

2 Preliminaries and notations

In this section, we give some basic definitions and properties of the fractional calculus theory which will be used further in this paper. For more details see [2]. For the finite interval $[a,b]$, we define the following fractional integral and derivatives.

Definition 2.1. *A real function $f(x)$, $x>0$, is said to be in the space C_μ, $\mu\in R$, if there exists a real number $(p>\mu)$, such that $f(x)=x^p f_1(x)$, where $f_1(x)\in C(0,\infty)$, and it is said to be in the space C_μ^m if $f^m\in C_\mu$, $m\in N$.*

Definition 2.2. *The Riemann- Liouville fractional integral operator of order $\alpha\geq 0$, of a function $f\in C_\mu$, $\mu\geq -1$, is defined as*

$$J^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (u-t)^{\alpha-1}f(t)\,\mathrm{d}t, \quad \alpha>0, \quad x>0, \quad J^0 f(x)=f(x). \tag{2.1}$$