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Quenching Time for a Semilinear Heat Equation with a Nonlinear Neumann Boundary Condition

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Abstract. In this paper we consider the finite time quenching behavior of solutions to a semilinear heat equation with a nonlinear Neumann boundary condition. Firstly, we establish conditions on nonlinear source and boundary to guarantee that the solution doesn't quench for all time. Secondly, we give sufficient conditions on data such that the solution quenches in finite time, and derive an upper bound of quenching time. Thirdly, under more restrictive conditions, we obtain a lower bound of quenching time. Finally, we give the exact bounds of quenching time of a special example.

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1 Introduction

In this paper, we mainly study the following initial-boundary value problem

$$\begin{cases} u_t = \Delta u + f(u), & x \in \Omega, \ t > 0, \\ \dfrac{\partial u}{\partial \nu} = g(u), & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

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where ν is the exterior normal vector of the $\partial\Omega$ assumed smooth enough, Ω is a star-shaped smoothly bounded domain in $\mathbb{R}^N$ ($N\geq 2$), and $u_0(x)$ is a positive bounded function satisfying the compatibility conditions. We say a solution u of the problem (1.1) quenches in finite time, *if $u>0$ exists in the classical sense for all $t\in[0,T)$, and satisfies*

$$\lim_{t\to T}\min_{x\in\overline{\Omega}} u(x,t)=0. \tag{1.2}$$

If quenching occurs, we denote the quenching time by T, or else $T=\infty$.

Quenching problems have been studied by many researchers (see [1–8] and the references therein), since the initial work of Kawarada [9] appeared in 1975. Contrary to quenching, another singularity is called blowup, we refer to [10–15] and the references therein for the latest results of blowup problems.

In [16], Fila and Levine studied the quenching phenomenon of the equation

$$\begin{cases} u_t=u_{xx}, & (x,t)\in(0,1)\times(0,T),\\ u_x(0)=0,\quad u_x(1)=-u^{-q}, & t\in(0,T),\\ u(x,0)=u_0(x)>0, & x\in[0,1], \end{cases} \tag{1.3}$$

and proved that quenching only occurs at $x=1$, and the quenching rate of the solution satisfies $c_1(T-t)^{\frac{1}{2(q+1)}}\leq u(1,t)\leq c_2(T-t)^{\frac{1}{2(q+1)}}$, where c_1, c_2 are positive constants.

Hu and Yin [17] studied the blowup profile near the blowup time for the heat equation

$$\begin{cases} u_t=\Delta u, & x\in\Omega,\ t>0,\\ \dfrac{\partial u}{\partial\nu}=u^p, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

where $p>1$. They proved that blowup occurs only on the boundary in finite time and the blowup rate is $\max_{x\in\overline{\Omega}}u(x,t)\sim c(T-t)^{-\frac{1}{2(p-1)}}$ if $u_0\geq 0$, and $\Delta u_0\geq 0$. For more generalized equations, we refer to [7].

Recently, Payne et al. [18–20] studied the blowup phenomena and derived the upper and lower bounds of blowup time of Eq. (1.1) under certain assumptions of f and g. As we know, many authors considered the rate estimates of the blowup or quenching solutions, and even blowup time estimate, but very few ones studied the exact estimate of quenching time. In this paper, under the different assumptions of f and g, we consider the quenching problem, and study the estimate of quenching time. First, we show the criteria for the solution u of (1.1) non-quenching and get the upper and lower bounds of quenching time of problem (1.1). Moreover, we give an example to show the applicability of our results. In the paper, we shall use the following Sobolev type inequality

$$\int_\Omega u^{\frac{3n}{2}}\mathrm{d}x\leq\left\{\frac{3}{2\rho_0}\int_\Omega u^n\mathrm{d}x+\frac{n}{2}\left(1+\frac{d}{\rho_0}\right)\int_\Omega u^{n-1}|\nabla u|\mathrm{d}x\right\}^{\frac{3}{2}},$$