JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **27**, No. 3, pp. 251-267

doi: 10.4208/jpde.v27.n3.6
September 2014

Strong Solutions for Nonhomogeneous Incompressible Viscous Heat-Conductive Fluids with Non-Newtonian Potential

MENG Qiu[1,*] and YUAN Hongjun[2]

[1] *Institute of Mathematics and Statistics, Beihua University, Jilin 132013, China.*
[2] *Institute of Mathematics, Jilin University, Changchun 130012, China.*

Received 18 February 2014; Accepted 13 August 2014

Abstract. We consider the Navier-Stokes system with non-Newtonian potential for heat-conducting incompressible fluids in a domain $\Omega\subset\mathcal{R}^3$. The viscosity, heat conduction coefficients and specific heat at constant volume are allowed to depend smoothly on the density and temperature. We prove the existence of unique local strong solutions for all initial data satisfying a natural compatibility condition. The difficult of this type model is mainly that the equations are coupled with elliptic, parabolic and hyperbolic, and the vacuum of density cause also much trouble, that is, the initial density need not be positive and may vanish in an open set.

AMS Subject Classifications: 35A05, 35D35, 76A05, 76D03

Chinese Library Classifications: O175.29

Key Words: Strong solutions; heat-conductive fluids; vacuum; Poincaré type inequality; non-Newtonian potential.

1 Introduction

The governing system of equations for a heat-conducting viscous fluids under the self-gravitational force and outer power can be described by the model of the fluids dynamic, that is, the incompressible full Navier-Stokes equations with non-Newtonian potential:

$$\partial_t\rho+\mathrm{div}(\rho u)=0, \tag{1.1}$$

$$\mathrm{div}u=0, \tag{1.2}$$

$$C_V(\partial_t(\rho\theta)+\mathrm{div}(\rho u\theta))-\mathrm{div}(\kappa\nabla\theta)=2\mu|du|^2+\rho h, \tag{1.3}$$

$$\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mathrm{div}(2\mu du)+\nabla P+\rho\nabla\Phi=\rho f, \tag{1.4}$$

$$\mathrm{div}[(|\nabla\Phi|^2+\varepsilon)^{\frac{p-2}{2}}\nabla\Phi]=4\pi g\Big(\rho-\frac{m_0}{|\Omega|}\Big),\qquad p>2, \tag{1.5}$$

*Corresponding author. *Email addresses:* mengqiu15@163.com (Q. Meng), hjy@jlu.edu.cn (H. Yuan)

in $(0,T)\times\Omega$ together with the boundary and the initial conditions

$$(u,\nabla\theta\cdot n,\Phi)=(0,0,0), \qquad \text{on } \partial\Omega\times(0,T), \tag{1.6}$$

$$(\rho,\rho\theta,\rho u)=(\rho_0,\rho_0\theta_0,\rho_0 u_0), \quad \text{in } \Omega. \tag{1.7}$$

Here we denote by ρ,θ,u,P,Φ,g and f the unknown density, temperature, velocity, pressure, non-Newtonian gravitational potential, gravitational constant and outer power, respectively, and h is a heat source. $du=\frac{1}{2}(\nabla u+\nabla^T u)$ is the deformation tensor. Eq. (1.5) may be formulated as the Euler-non-Newtonian Poisson equation. We assume that viscosity coefficient $\mu=\mu(\rho,\theta)$, specific heat at constant volume $C_V=C_V(\rho,\theta)$ and heat conductivity $\kappa=\kappa(\rho,\theta)$ are positive functions of ρ and θ. Finally, $(0,T)\times\Omega$ is the time-space domain for the evolution of the fluid, where T is a finite positive constant and Ω is a bounded domain with smooth boundary $\partial\Omega$ whose unit outward normal is n.

For $\Phi=0$, the problem has been studied by many authors [1–8]. Very recently, Cho and Kim [9] showed that the problem has a unique local solution (ρ,u,P,θ) with the main hypothesis

$$\inf\rho_0=0, \quad \text{in } \Omega, \tag{1.8}$$

and some natural compatibility conditions:

$$\begin{aligned} &-\mathrm{div}(2\mu_0 du_0)+\nabla P_0=\rho_0^{\frac{1}{2}}g_1,\\ &-\mathrm{div}(\kappa_0\nabla\theta_0)-2\mu_0|du_0|^2=\rho_0^{\frac{1}{2}}g_2, \end{aligned} \tag{1.9}$$

for some $P_0\in H^1(\Omega)$ and functions $(g_1,g_2)\in L^2(\Omega)$. And further they assume an additional condition, such that

$$0<\mu,C_V,\kappa\in C^1(\mathcal{R}^2),\ \ \mu=\mu(\rho,\rho\theta),\ \ C_V=C_V(\rho,\rho\theta),\ \ \kappa=\kappa(\rho,\rho\theta), \tag{1.10}$$

for nonconstant coefficients.

The aim of this paper is to use the method of [9] to prove the existence of unique local strong solutions to (1.1)-(1.7) with $\inf\rho_0=0$. Here it should be noted that, in [9], the authors prescribed the homogeneous Dirichlet boundary condition for the temperature θ, i.e. $\theta|_{\partial\Omega}=0$, instead of the homogeneous Neumann boundary condition, i.e. $\nabla\theta\cdot n|_{\partial\Omega}=0$, for technical reasons, we will use a Poincaré type inequality [10, 11] to circumvent this difficulty.

The following is our MAIN RESULT.

Theorem 1.1. *Assume that the data $(\rho_0,u_0,\theta_0,h,f)$ satisfy the regularity condition*

$$\begin{aligned} &\rho_0\ge 0, \quad \rho_0\in W^{1,q}(\Omega), \quad \theta_0\in H^2(\Omega), \quad \nabla\theta_0\cdot n|_{\partial\Omega}=0,\\ &u_0\in H^2(\Omega)\cap H_0^1(\Omega), \quad \mathrm{div}u_0=0, \quad u_0|_{\partial\Omega}=0,\\ &(h,f)\in C([0,T];L^2(\Omega))\cap L^2(0,T;L^q(\Omega)), \quad (h_t,f_t)\in L^2(0,T;H^{-1}(\Omega)), \end{aligned}$$