JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **27**, No. 4, pp. 318-332

doi: 10.4208/jpde.v27.n4.3
December 2014

Existence and Boundary Behavior of Positive Solutions of Quasi-linear Elliptic Singular Equations with a Gradient term

YUAN Junli[1] and YANG Zuodong[2,3]*

[1] *School of Science, Nantong University, Nantong 226006, China.*
[2] *Institute of Mathematics, School of Mathematics Science, Nanjing Normal University, Nanjing 210023, China.*
[3] *School of Teacher Education, Nanjing Normal University, Nanjing 210097, China.*

Received 25 May 2014; Accepted 17 October 2014

Abstract. Existence of positive solutions of a class of quasi-linear elliptic equation with a gradient term is obtained by using super-solution and sub-solution method. In particular, we study the asymptotic behavior of the solution near the boundary up to the second order under various assumptions on the growth of the coefficients of the equation. The results of this paper is new and extend previously known results.

AMS Subject Classifications: 35J65, 35B25

Chinese Library Classifications: O175.8, O175.25, O175.29

Key Words: Existence; quasi-linear elliptic equation; super-solution; sub-solution.

1 Introduction

In this paper, we consider positive solutions of the following problem

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u)+g(u)|\nabla u|^{q}+f(u)=0, \quad x\in D, \quad u=0, \quad \text{on } \partial D, \tag{1.1}$$

where D is a bounded smooth domain in $\mathbf{R}^N$, $q\in(0,p)$, $g(t)$ is a decreasing smooth function in $(0,\infty)$ (which can be unbounded) and $f(t)$ is a decreasing and positive smooth function in $(0,\infty)$, which approaches infinity as $t\to 0$.

Equations of the above form are mathematical models occurring in studies of the p-Laplace equation, generalized reaction-diffusion theory, non-Newtonian fluid theory([1,2]), non-Newtonian filtration([3]) and the turbulent flow of a gas in a porous medium([4]).

*Corresponding author. *Email addresses:* `zdyang_jin@263.net` (Z. Yang)

In the non-Newtonian fluid theory, the quantity p is characteristic of the medium. Media with $p>2$ are called dilatant fluids and those with $p<2$ are called pseudo-plastics. If $p=2$, they are Newtonian fluids. When $p\neq 2$, the problem becomes more complicated since certain nice properties inherent to the case $p=2$ seem to be lost or at least difficult to verify. The main differences between $p=2$ and $p\neq 2$ can be found in [5,6].

By a positive entire solution of (1.1) we mean a function $u\in W_0^{1,p}(D)\cap C^1(D)$ which satisfies (1.1) at every point of D in a weak sense with $u>0$ in D, that is $u\in W_0^{1,p}(D)\cap C^1(D)$ which satisfies

$$-\int_D |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi \mathrm{d}x+\int_D g(u)|\nabla u|^q\varphi \mathrm{d}x+\int_D f(u)\varphi \mathrm{d}x=0, \qquad \forall\varphi\in C_0^\infty(D),$$

and $u>0$ in D. The notions of a super-solution and a sub-solution are defined similarly, by replacing "$=$" with "$\geq$" and "$\leq$", respectively.

Equations of the type (1.1) have been studied by many authors When $p=2$, see, for instance, [7–17] and references therein. The homogeneous Dirichlet problem for the equation

$$\Delta u+p(x)u^{-\gamma}=0, \quad \text{in } D,$$

was studied in [7] and in [15], where existence and uniqueness of a positive solution $u(x)$ continuous to boundary was proved for $\gamma>0$, and $p(x)$ is positive and bounded in $\overline{D}$.

In recent years, several authors investigated the existence of bounded positive solutions of (1.1) obtained many results in [18–30] and references therein when $f:\mathbf{R}\to\mathbf{R}$ is locally Hölder continuous. In [21], we study the existence of positive bounded solution of the following

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u)+f(x,u)+g(|x|)|x\cdot\nabla u|^{p-2}(x\cdot\nabla u)=0, \quad x\in G_R,$$

where $G_R=\{x\in\mathbf{R}^N:|x|>R\,(R>0)\}$, $N\geq 1$, $1<p<N$, $f:G_R\times\mathbf{R}\to\mathbf{R}$is locally Hölder continuous, $g\in C^1([0,+\infty),\mathbf{R})$. However, it seems that very little is known about the existence of positive solutions for singular p-Laplacian equation except for the work by Yang and Miao [31]. The existence of positive entire solutions of singular p-Laplacian equation

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u)+f(x,u,\nabla u)u^{-\gamma}=0, \quad x\in\mathbf{R}^N,$$

was studied in [31]. In this paper, we will consider a class of more general function which contains $u^{-\gamma}$.

Motivated by the results of the above cited papers, we obtain an existence and asymptotic behavior result of positive solutions of (1.1). When $p=2$, the related results have been obtained in [14]. Our theorem complements for existence, asymptotic behavior and extends the results in [7,14,15,21–23,31].

We point out that our methods are similar to those employed in [13,14,32] as regards to nonlinear elliptic problems. In Section 2, we study the existence and asymptotic behavior of radial solutions of (1.1) in balls. In Section 3, we derive estimates in a general domain by a comparison argument.