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An Existence Result for a Class of Chemically Reacting Systems with Sign-Changing Weights

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Abstract. We prove the existence of positive solutions for the system

$$\begin{cases} -\Delta_p u = \lambda a(x) f(v) u^{-\alpha}, & x \in \Omega, \\ -\Delta_q v = \lambda b(x) g(u) v^{-\beta}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases}$$

where $\Delta_r z = \mathrm{div}(|\nabla z|^{r-2}\nabla z)$, for $r > 1$ denotes the r-Laplacian operator and λ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$ with sufficiently smooth boundary and $\alpha,\beta \in (0,1)$. Here $a(x)$ and $b(x)$ are C^1 sign-changing functions that maybe negative near the boundary and f,g are C^1 nondecreasing functions, such that $f,g\colon [0,\infty) \to [0,\infty)$; $f(s) > 0$, $g(s) > 0$ for $s > 0$, $\lim_{s\to\infty} g(s) = \infty$ and

$$\lim_{s\to\infty} \frac{f(Mg(s)^{\frac{1}{q-1}})}{s^{p-1+\alpha}} = 0, \qquad \forall M > 0.$$

We discuss the existence of positive weak solutions when f, g, $a(x)$ and $b(x)$ satisfy certain additional conditions. We employ the method of sub-supersolution to obtain our results.

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1 Introduction

In this paper, we consider the existence of positive weak solutions for the nonlinear singular system

$$\begin{cases} -\Delta_p u = \lambda a(x) \frac{f(v)}{u^{\alpha}}, & x \in \Omega, \\ -\Delta_q v = \lambda b(x) \frac{g(u)}{v^{\beta}}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

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where $\Delta_r z = \text{div}(|\nabla z|^{r-2}\nabla z)$, for $r>1$ denotes the r-Laplacian operator and Ω is a bounded domain in $\mathbb{R}^n$, $n\geq 1$ with smooth boundary, $\alpha,\beta\in(0,1)$. Here $a(x)$ and $b(x)$ are C^1 sing-changing functions that maybe negative near the boundary and f,g are C^1 nondecreasing functions such that $f,g:[0,\infty)\to[0,\infty)$; $f(s)>0$, $g(s)>0$ for $s>0$.

Systems of singular equations like (1.1) are the stationary counterpart of general evolutionary problems of the form

$$\begin{cases} u_t = \eta\Delta_p u + \lambda\frac{f(v)}{u^\alpha}, & x\in\Omega, \\ v_t = \delta\Delta_q v + \lambda\frac{g(u)}{v^\beta}, & x\in\Omega, \\ u = v = 0, & x\in\partial\Omega, \end{cases}$$

where η and δ are positive parameters. This system is motivated by an interesting applications in chemically reacting systems, where u represents the density of an activator chemical substance and v is an inhibitor. The slow diffusion of u and the fast diffusion of v is translated into the fact that η is small and δ is large (see [1]).

Recently, such problems have been studied in [2–4]. Also in [2], the authors have studied the existence results for the system (1.1) in the case $a\equiv 1$, $b\equiv 1$. Here we focus on further extending the study in [2] to the system (1.1). In fact, we study the existence of positive solution to the system (1) with sign-changing weight functions $a(x)$,$b(x)$. Due to these weight functions, the extensions are challenging and nontrivial. Our approach is based on the method of sub-super solutions (see [5–7]).

To precisely state our existence result we consider the eigenvalue problem

$$\begin{cases} -\Delta_r\phi = \lambda|\phi|^{r-2}\phi, & x\in\Omega, \\ \phi = 0, & x\in\partial\Omega. \end{cases} \tag{1.2}$$

Let $\phi_{1,r}$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (1.2) such that $\phi_{1,r}(x)>0$ in Ω, and $\|\phi_{1,r}\|_\infty = 1$ for $r=p,q$. Let $m,\sigma,\delta>0$ be such that

$$\sigma\leq\phi_{1,r}\leq 1, \quad x\in\Omega-\overline{\Omega_\delta}, \tag{1.3}$$

$$\lambda_{1,r}\phi_{1,r}^r - \left(1-\frac{sr}{r-1+s}\right)|\nabla\phi_{1,r}|^r \leq -m, \quad x\in\overline{\Omega_\delta}, \tag{1.4}$$

for $r=p,q$, and $s=\alpha,\beta$, where $\overline{\Omega_\delta}:=\{x\in\Omega | d(x,\partial\Omega)\leq\delta\}$. (This is possible since $|\nabla\phi_{1,r}|^r\neq 0$ on $\partial\Omega$ while $\phi_{1,r}=0$ on $\partial\Omega$ for $r=p,q$. We will also consider the unique solution $\zeta_r\in W_0^{1,r}(\Omega)$ of the boundary value problem

$$\begin{cases} -\Delta_r\zeta_r = 1, & x\in\Omega, \\ \zeta_r = 0, & x\in\partial\Omega, \end{cases}$$

to discuss our existence result, it is known that $\zeta_r>0$ in Ω and $\partial\zeta_r/\partial n<0$ on $\partial\Omega$.