JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 1, pp. 74-94

doi: 10.4208/jpde.v28.n1.7
March 2015

Classification of Solutions to a Critically Nonlinear System of Elliptic Equations on Euclidean Half-Space

GLUCK Mathew R.[1,*] and ZHANG Lei [2]

[1] *Department of Mathematical Sciences, University of Alabama in Huntsville, Huntsville, AL 35899.*
[2] *Department of Mathematics, University of Florida, Gainesville, FL 32611-8105.*

Received 13 January 2014; Accepted 25 December 2014

Abstract. For $N\geq 3$ and non-negative real numbers a_{ij} and b_{ij} $(i,j=1,\cdots,m)$, the semi-linear elliptic system

$$\begin{cases}\Delta u_i+\prod\limits_{j=1}^{m}u_j^{a_{ij}}=0, & \text{in } \mathbb{R}_+^N,\\ \dfrac{\partial u_i}{\partial y_N}=c_i\prod\limits_{j=1}^{m}u_j^{b_{ij}}, & \text{on } \partial\mathbb{R}_+^N,\end{cases}\qquad i=1,\cdots,m,$$

is considered, where $\mathbb{R}_+^N$ is the upper half of N-dimensional Euclidean space. Under suitable assumptions on the exponents a_{ij} and b_{ij}, a classification theorem for the positive $C^2(\mathbb{R}_+^N)\cap C^1(\overline{R_+^N})$-solutions of this system is proven.

AMS Subject Classifications: 35J57, 35J66, 35K57
Chinese Library Classifications: O175.25
Key Words: Nonlinear elliptic systems.

1 Introduction

Let $N\geq 3$ be a positive integer and let $\mathbb{R}_+^N=\{(y_1,\cdots,y_N)\in\mathbb{R}^N : y_N>0\}$ denote the upper half of N-dimensional Euclidean space. Fix a positive integer m and set $J=\{1,\cdots,m\}$. Let $\mathcal{A}=[a_{ij}]$ be an $m\times m$ matrix with nonnegative entries. We are concerned with the classical solutions of the semi-linear elliptic system

$$\Delta u_i+\prod_{j=1}^{m}u_j^{a_{ij}}=0,\qquad \text{in } \Omega\subset\mathbb{R}^N \text{ for all } i\in J. \tag{1.1}$$

*Corresponding author. *Email addresses:* `mathew.gluck@uah.edu` (M. Gluck), `leizhang@ufl.edu` (L. Zhang)

This system and its variants have been studied extensively in numerous contexts. For example, (1.1) arises as the system of equations for a steady-state solution to the corresponding parabolic reaction-diffusion system. In particular, when $m=2$ the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \Delta u_1 + u_1^{a_{11}} u_2^{a_{12}}, & \text{for } y \in \Omega, t>0, \\ \dfrac{\partial u_2}{\partial t} = \Delta u_2 + u_1^{a_{21}} u_2^{a_{22}}, & \text{for } y \in \Omega, t>0, \end{cases} \tag{1.2}$$

has received much attention. For example, when $a_{11}=a_{22}=0$, (1.2) gives a simple model for heat propagation in a two-component combustible mixture [1]. Variants of (1.2) have also been used to model the diffusing densities of two biological species when each specie finds its subsidence from the activity of the other specie [2]. It is well-known that a thorough understanding of (1.1) is highly beneficial to obtaining an understanding of (1.2). For example, under appropriate assumptions on $\mathcal{A}$, in [3] and [4] Mitidieri proved nonexistence results for (1.1) when $\Omega=\mathbb{R}^N$ and $m=2$. These results were refined by Zheng in [5] and then used to derive blow-up (in time) estimates for solutions of (1.2) that satisfy suitable initial and boundary conditions. For more results concerning these parabolic systems and their variants the reader is referred to [6, 7] and the references therein.

An interesting case of (1.1) arises when $\mathcal{A}$ satisfies

$$\begin{cases} a_{ij} \geq 0, & \text{for all } (i,j) \in J \times J, \\ \mathcal{A} \text{ is irreducible }, & \\ \sum\limits_{j=1}^{m} a_{ij} = \frac{N+2}{N-2}, & \text{for all } i \in J. \end{cases} \tag{1.3}$$

Recall that an $m \times m$-matrix $\mathcal{A}$ is called *irreducible* if there is no partition $J = I_1 \cup I_2$ such that $a_{ij}=0$ for all $i \in I_1$, and $j \in I_2$. When $m=1$ equations (1.1) reduce to

$$\Delta u + K u^{(N+2)/(N-2)} = 0, \tag{1.4}$$

with $K=1$. Eq. (1.4) has been studied extensively as it arises in relation to the famous Yamabe problem. The Yamabe problem asks whether it is always possible to conformally deform the metric g of a given smooth compact Riemannian manifold to a metric $\hat{g} = u^{4/(N-2)} g$ whose scalar curvature is constant. Through the works of Trudinger [8], Aubin [9] and Schoen [10], the Yamabe problem was proven affirmative. See [11] and the references therein for results regarding the Yamabe problem. For $\mathcal{A}$ satisfying (1.3) and $\Omega=\mathbb{R}^N$, the classical solutions of (1.1) were classified by Chipot, Shafrir and Wolansky in [12] (see also [13]). Their result is the following.

Theorem 1.1 (Chipot, Shafrir and Wolansky [12]). *Suppose $\mathcal{A}$ satisfies* (1.3). *If $u_1, \cdots, u_m$ are positive $C^2(\mathbb{R}^N)$-solutions of* (1.1) *with $\Omega=\mathbb{R}^N$ then*

$$u_i(y) = \frac{\beta_i}{\left(\sigma^2 + |y-y^0|^2\right)^{(N-2)/2}}, \qquad \text{for all } i \in J, \tag{1.5}$$