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A Remark on the Level Sets of the Graph of Harmonic Functions Bounded by Two Circles in Parallel Planes

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Abstract. In this paper, we find two auxiliary functions and make use of the maximum principle to study the level sets of harmonic function defined on a convex ring with homogeneous Dirichlet boundary conditions in $\mathbb{R}^2$. In higher dimensions, we also have a similar result to Jagy's.

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1 Introduction

The geometry of the level sets of the solutions of elliptic partial differential equations is a classical subject. For instance, Ahlfors [1] contains the well-known result that level curves of the Green function on a simply connected convex domain in the plane are convex Jordan curves. Motivated by catenoid or the "Riemann Stair-case" in minimal surface theory, in 1956, Shiffman [2] proved the following two results: (i) if a minimal surface S in R^3 is bounded by convex curves in parallel planes, and S is topologically an annulus, then the intersections of S with all other parallel planes are also convex curves; (ii) if the boundaries are circles in parallel planes, then the intersections of S with all other parallel planes are also circles. In 1957, Gabriel [3] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex. Later, in 1977, Lewis [4] extended Gabriel's result to p-harmonic functions in higher dimensions. In 1990, Jagy [5]

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got the high dimension generalization of the second result of Shiffman [2]. Up to now, the high dimension generalization of the first result of Shiffman [2] is still open. From Korevaar [6], we know that it is true for the graph case.

Shiffman used complex analysis to prove the above result. In the proof of case (i), Shiffman introduced a curvature type function ψ_u (in page 80 [2]) of the level curve of the minimal surfaces. He proved that ψ_u is a harmonic function and it is non-negative if the level curves are convex. In the proof of case (ii), Shiffman [2] introduced an important auxiliary function, now named the Shiffman function, which is a harmonic function on the minimal surface.

Remark 1.1. In order to generalize the Shiffman's curvature type function ψ_u (see page 80 in [2]) from minimal surface to harmonic function, Talenti [7] got the following result. Suppose u is harmonic and has no critical points in a domain $\Omega\subset\mathbb{R}^2$, K is the curvature of the level curves of u with respect to the normal direction ∇u, defined as the following

$$K=\sum_{i,j=1}^{2}\frac{\partial \det(u_{ij})}{\partial u_{ij}}u_iu_j|\nabla u|^{-3}, \tag{1.1}$$

then $K/|\nabla u|$ is a harmonic function in Ω. More recently, Ma-Ou-Zhang [8] had generalized the above result to high dimension harmonic function.

In this paper, for the harmonic function defined on plane domain, we find two harmonic functions corresponding to Shiffman function. Now we state our main result.

Theorem 1.1. *Let u satisfy*

$$\begin{cases} \Delta u=0 & \text{in} \quad \Omega=\Omega_0\backslash\overline{\Omega}_1, \\ u=0 & \text{on} \quad \partial\Omega_0, \\ u=1 & \text{on} \quad \partial\Omega_1, \end{cases} \tag{1.2}$$

where Ω_0 and Ω_1 are bounded smooth convex domains in $\mathbb{R}^2$, and $\overline{\Omega}_1\subset\Omega_0$, K is the curvature of the level curves of u defined as in (1.1). *Then*

(i) $\varphi=|\nabla u|^{-3}(K_1u_2-K_2u_1)$ *is a harmonic function in* Ω.

(ii) $\psi=|\nabla u|^{-2}K^{-1}(K_1u_2-K_2u_1)$ *satisfies the following equation*

$$\Delta\psi+\sum_{i=1}^{2}b_i\psi_i=0, \quad \text{in} \quad \Omega,$$

where $b_i\,(i=1,2)$ are bounded continuous functions.

Corollary 1.1. *As a consequence of Theorem* 1.1, *since $(K_1u_2-K_2u_1)$ is the tangential derivative of the curvature K of the level sets of u on its level set, from the maximum principle, we have if $\partial\Omega_0$ and $\partial\Omega_1$ are circles, then the intermediate cross-sections must be circles.*