Renormalized Solutions of Nonlinear Parabolic Equations in Weighed Variable-Exponent Space

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Abstract. This article is devoted to study the existence of renormalized solutions for the nonlinear $p(x)$-parabolic problem in the Weighted-Variable-Exponent Sobolev spaces, without the sign condition and the coercivity condition.

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1 Introduction

In the present paper we establish the existence of renormalized solutions for a class of nonlinear $p(x)$-parabolic equation of the type:

\[
(P) \quad \begin{cases}
\frac{\partial u}{\partial t} - \text{div}(a(x,t,u,\nabla u)) + H(x,t,u,\nabla u) = f, & \text{in } Q = \Omega \times (0,T), \\
u(x,0) = u_0, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega \times (0,T).
\end{cases}
\]

In the problem $(P)$, $\Omega$ is a bounded domain in $\mathbb{R}^N, N \geq 1$, $T$ is a positive real number, while $u_0 \in L^1(\Omega)$, $f \in L^1(Q)$.

The operator $-\text{div}(a(x,t,u,\nabla u)$ is a Weighted Leray–Lions operator defined on $L^{p_+}(0,T; W^{1,p(x)}(\Omega;\omega))$ (see assumptions (3.1)-(3.3) of Section 3) which is coercive and where $H$ is a nonlinear lower order term, satisfying some growth condition but no sign condition or

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the coercivity condition and the critical growth condition on $H$ is with respect to $\nabla u$ and no growth with respect to $u$. For the degenerated parabolic equations the existence of weak solutions have been proved by Aharouch et al. [1] in the case where $a$ is strictly monotone, $H = 0$ and $f \in L^{p'}(0,T;W^{-1,p'}_0(\Omega,\omega^*))$ and the problem $(P)$ is studied by Akdim et al. [2] in the degenerated Weighted Sobolev space with $u = b(x,u)$. This notion was adapted to the study with $H = 0$ of some nonlinear elliptic problems on Sobolev spaces a exponent variable with Dirichlet boundary conditions by Boccardo et al. [3] and Lions [4]. In the case where $p(x)$ is a constant, some results have been proved by Akdim et al. [5]. Recently, while $a(x,t,u,\nabla u) = |\nabla u|^{p(x) - 2}\nabla u$, Zhang and Zhou [6] proved the existence of a renormalised and entropy solutions with $L^1$-data, see also Bendahmane et al. [7].

This paper is organized as follows. In Section 2, we state some basic results for the weighted variable exponent Lebesgue–Sobolev spaces which is given in [8]. In Section 3, we give our basic assumption and the definition of a renormalized solution of the problem $(P)$ for which our problem has a solution. In Section 4, we establish the existence of such a solution in Theorem 4.1. In Section 5, we give the proof of Theorem 4.2, Lemma 4.2 and Proposition 4.2 (see appendix).

2 Preliminaries

In this section, we state some elementary properties for the Weighted Variable Exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega,\omega)$ which will be used in the next sections. The basic properties of the variable exponent Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$, that is, when $\omega(x) \equiv 1$ can be found from [9, 10].

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$. Set

$$C_+ (\Omega) = \{ p \in C(\Omega) : \min_{x \in \Omega} p(x) > 1 \}.$$ 

For any $p \in C_+ (\Omega)$, we define

$$p^+ = \max_{x \in \Omega} p(x), \quad p^- = \min_{x \in \Omega} p(x).$$

For any $p \in C_+ (\Omega)$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions $u$ such that

$$L^{p(x)}(\Omega,\omega) = \left\{ u : \Omega \to \mathbb{R}, \text{measurable}, \int_{\Omega} |u(x)|^{p(x)} \omega(x) dx < \infty \right\}.$$ 

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \omega(x) dx \leq 1 \right\}$$