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Renormalized Solutions of Nonlinear Parabolic Equations in Weigthed Variable-Exponent Space

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Abstract. This article is devoted to study the existence of renormalized solutions for the nonlinear $p(x)$-parabolic problem in the Weighted-Variable-Exponent Sobolev spaces, without the sign condition and the coercivity condition.

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1 Introduction

In the present paper we establish the existence of renormalized solutions for a class of nonlinear $p(x)$-parabolic equation of the type:

$$(\mathcal{P})\quad \begin{cases} \dfrac{\partial u}{\partial t}-\operatorname{div}(a(x,t,u,\nabla u))+H(x,t,u,\nabla u)=f, & \text{in } Q=\Omega\times(0,T),\\ u(x,0)=u_0, \quad \text{in } \Omega, & \\ u=0, \quad \text{on } \partial\Omega\times(0,T). & \end{cases}$$

In the problem $(\mathcal{P})$, Ω is a bounded domain in $\mathbb{R}^N, N\geq 1$, T is a positive real number, while $u_0\in L^1(\Omega)$, $f\in L^1(Q)$.

The operator -div$(a(x,t,u,\nabla u)$ is a Weighted Leray–Lions operator defined on $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega,\omega))$ (see assumptions (3.1)-(3.3) of Section 3) which is coercive and where H is a nonlinear lower order term, satisfying some growth condition but no sign condition or

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the coercivity condition and the critical growth condition on H is with respect to ∇u and no growth with respect to u.

For the degenerated parabolic equations the existence of weak solutions have been proved by Aharouch et al. [1] in the case where a is strictly monotone, $H=0$ and $f\in L^{p'}(0,T;W_0^{-1,p'}(\Omega,\omega^*))$ and the problem $(\mathcal{P})$ is studied by Akdim et al. [2] in the degenerated Weighted Sobolev space with $u=b(x,u)$.

This notion was adapted to the study with $H=0$ of some nonlinear elliptic problems on Sobolev spaces a exponent variable with Dirichlet boundary conditions by Boccardo et al. [3] and Lions [4]. In the case where $p(x)$ is a constant, some results have been proved by Akdim et al. [5].

Recently, while $a(x,t,u,\nabla u)=|\nabla u|^{p(x)-2}\nabla u$, Zhang and Zhou [6] proved the existence of a renomalised and entropy solutions with L^1-data, see also Bendahmane et al. [7].

This paper is organized as follows. In Section 2, we state some basic results for the weighted variable exponent Lebesgue-Sobolev spaces which is given in [8]. In Section 3, we give our basic assumption and the definition of a renormalized solution of the problem (P) for which our problem has a solution. In Section 4, we establish the existence of such a solution in Theorem 4.1. In Section 5, we give the proof of Theorem 4.2, Lemma 4.2 and Proposition 4.2 (see appendix).

2 Preliminaries

In this section, we state some elementary properties for the Weighted Variable Exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega,\omega)$ which will be used in the next sections. The basic properties of the variable exponent Lebesgue–Sobolev spaces $W^{1,p(x)}(\Omega,\omega)$, that is, when $\omega(x)\equiv 1$ can be found from [9,10].

Let Ω be a bounded open subset of $\mathbb{R}^N(N\geq 2)$. Set

$$C_+(\overline{\Omega})=\{p\in C(\overline{\Omega}):\min_{x\in\overline{\Omega}}p(x)>1\}.$$

For any $p\in C_+(\overline{\Omega})$, we define

$$p^+=\max_{x\in\overline{\Omega}}p(x),\quad p^-=\min_{x\in\overline{\Omega}}p(x).$$

For any $p\in C_+(\overline{\Omega})$, we introduce the weighted variable exponent Lebesgue space $L^{p(x)}(\Omega,\omega)$ that consists of all measurable real-valued functions u such that

$$L^{p(x)}(\Omega,\omega)=\left\{u:\Omega\to\mathbb{R},\text{ measurable, }\int_\Omega|u(x)|^{p(x)}\omega(x)\mathrm{d}x<\infty\right\}.$$

Then, $L^{p(x)}(\Omega,\omega)$ endowed with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega,\omega)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\omega(x)\mathrm{d}x\leq 1\right\}$$