Remarks on Liouville Type Result for the 3D Hall-MHD System

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Abstract. In this paper, we consider the 3D Hall-MHD system, and provide an improved Liouville type result for its stationary version.

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1 Introduction

This paper concerns itself with the three-dimensional (3D) Hall-magnetohydrodynamics system (Hall-MHD):

\[
\begin{align*}
\begin{cases}
    u_t + (u \cdot \nabla) u + \nabla \pi &= (\nabla \times B) \times B + \Delta u, \\
    \nabla \cdot u &= 0, \\
    B_t - \nabla \times (u \times B) + \nabla \times [(\nabla \times B) \times B] &= \Delta B,
\end{cases}
\end{align*}
\]

in \( \mathbb{R}^3 \times (0, \infty) \), \( (1.1) \)

where \( u \) is fluid velocity field, \( B \) is the magnetic field, and \( \pi \) is a scalar pressure. We prescribe the initial data to satisfy the condition

\[
\nabla \cdot u(0) = \nabla \cdot B(0) = 0.
\]

The first systematic study of the Hall-MHD system is pioneered by Lighthill [1] followed by Campos [2]. Comparing with the usual MHD equations, the Hall-MHD system
has the Hall term $\nabla \times [\nabla \times B] \times B$ in (1.1)$_2$, which may become significant for such problems as magnetic reconnection in geo-dynamo [3], star formation [4, 5], neutron stars [6] and space plasmas [7, 8].

Mathematically, the Hall-MHD system can be derived from either two-fluids or kinetic models (see [3]), and the global existence of weak solutions, local existence and uniqueness of smooth solutions, blow-up criteria and small data global existence of classical solutions were established in [9, 10]. For the fractional Hall-MHD, the reader is referred to [11].

The stationary version of (1.1) is

$$
\begin{cases}
(u \cdot \nabla)u + \nabla \pi = (\nabla \times B) \times B + \Delta u, \\
\nabla \cdot u = 0,
\end{cases}
$$

(1.3)

And in [9], the authors established the following Liouville type theorem.

**Theorem 1.1.** ([9]) Let $u, B$ be $C^2(\mathbb{R}^3)$ solutions to (1.3) satisfying

$$
u, B \in L^\infty(\mathbb{R}^3) \cap L^2(\mathbb{R}^3); \quad \nabla u, \nabla B \in L^2(\mathbb{R}^3).$$

Then we have $u = B = 0$.

It is not natural to assume that the boundedness of the solution $u, B$ (see [12, 13]), and the aim of this paper is to improving Theorem 1.1 as

**Theorem 1.2.** Let $u, B$ be $C^2(\mathbb{R}^3)$ solutions to (1.3) satisfying

$$u, B \in L^2(\mathbb{R}^3); \quad \nabla u, \nabla B \in L^2(\mathbb{R}^3).$$

Then we have $u = B = 0$.

### 2 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2.

We first derive an estimate of the pressure. Taking the divergence of (1.3)$_1$, and using the vector identity

$$(\nabla \times B) \times B = -\nabla \frac{|B|^2}{2} + (B \cdot \nabla) B,$$

we obtain

$$-\Delta \pi = \sum_{i,j=1}^3 \partial_i \partial_j (u_i u_j - B_i B_j) + \Delta \frac{|B|^2}{2}.$$  

Classical elliptic regularity results then yields

$$\|\pi\|_{L^q} \leq C \|(u, B)\|_{L^2}^2, \quad 1 < q < \infty.$$  

(2.1)