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# A Logarithmically Improved Blow-up Criterion for a Simplified Ericksen-Leslie System Modeling the Liquid Crystal Flows

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**Abstract.** In this paper, we prove a logarithmically improved blow-up criterion in terms of the homogeneous Besov spaces for a simplified 3D Ericksen-Leslie system modeling the hydrodynamic flow of nematic liquid crystal. The result shows that if a local smooth solution $(u,d)$ satisfies

$$\int_0^T \frac{\|u\|_{\dot{B}^{-r}_{\infty,\infty}}^{\frac{2}{1-r}}+\|\nabla d\|_{L^\infty}^2}{1+\ln(e+\|u\|_{H^s}+\|\nabla d\|_{H^s})}\mathrm{d}t<\infty,$$

with $0\leq r<1$ and $s\geq 3$, then the solution $(u,d)$ can be smoothly extended beyond the time $T$.

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**Key Words**: Ericksen-Leslie system; Navier-Stokes equations; blow-up criterion.

## 1 Introduction

Liquid crystal, which is a state of matter capable of flow, but its molecules may be oriented in a crystal-like way. Liquid crystal exhibits a phase of matter that has properties between those of a conventional liquid and those of a solid crystal, thus it is commonly

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regarded as the fourth state of matter, different from gases, liquid, and solid. Nowadays, three main types of liquid crystals are distinguished, nematic, termed smectic and cholesteric. Among these three types of liquid crystals, the nematic phase appears to be the most common one, and is aggregations of molecules which possess same orientational order and are made of elongated, rod-like molecules. Since the nematic liquid crystal materials have remarkable applications in various technological fields, there have been numerous attempts to formulate continuum theories in describing the behavior of liquid crystal flows, we refer the readers to Ericksen [1] and Leslie [2].

In this paper, we study the following simplified 3D Ericksen-Leslie system modeling the hydrodynamic flow of nematic liquid crystal materials:

$$\partial_t u-\Delta u+(u\cdot\nabla)u+\nabla P=-\nabla\cdot(\nabla d\odot\nabla d), \quad \text{in } \mathbb{R}^3\times(0,+\infty), \tag{1.1}$$

$$\partial_t d+(u\cdot\nabla)d=\Delta d+|\nabla d|^2 d, \quad \text{in } \mathbb{R}^3\times(0,+\infty), \tag{1.2}$$

$$\nabla\cdot u=0, \quad \text{in } \mathbb{R}^3\times(0,+\infty), \tag{1.3}$$

$$(u,d)|_{t=0}=(u_0,d_0), \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

where $u(x,t):\mathbb{R}^3\times(0,+\infty)\to\mathbb{R}^3$ is the unknown velocity field of the flow, $P(x,t):\mathbb{R}^3\times(0,+\infty)\to\mathbb{R}$ is the scalar pressure and $d:\mathbb{R}^3\times(0,+\infty)\to\mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$) is the unknown (averaged) macroscopic/continuum molecule orientation of the nematic liquid crystal flow, $\nabla\cdot u=0$ represents the incompressible condition, $u_0$ is a given initial velocity field with $\nabla\cdot u_0=0$ in the distributional sense, $d_0:\mathbb{R}^3\to\mathbb{S}^2$ is a given initial liquid crystal orientation field. The notation $\nabla d\odot\nabla d$ denotes the $n\times n$ matrix whose $(i,j)$-th entry is given by $\partial_i d\cdot\partial_j d$ $(1\le i,j\le 3)$, and hence

$$\nabla\cdot(\nabla d\odot\nabla d)=\sum_{k=1}^{3}\Delta d_k\nabla d_k+\frac{1}{2}\nabla|\nabla d|^2.$$

The above simplified Ericksen-Leslie system was firstly proposed by Lin [3] back in the late 1980s. It is a macroscopic continuum description of the time evolution of the rod-like liquid crystal materials under the influence of both the velocity field $u$ and the macroscopic description of the microscopic orientation configurations $d$. Mathematical analysis of the system (1.1)–(1.4) was initially studied by a series of papers by Lin and Liu [4, 5]. Later on, there are considerably extensive studies devoted to global existence of weak solutions, global strong solutions with small initial data, uniqueness and regularity criteria of weak solutions, blow-up criteria of local smooth solutions and other related topics, we refer the readers to see, e.g., [6–11] and the references therein.

In the present paper, we are interested in the short time classical solution to the system (1.1)–(1.4) and address a blow-up criterion that characterizes the first finite singular time of the local classical solution. Since the strong solutions of the heat flow of harmonic maps, i.e., the case $u\equiv 0$ in (1.1)–(1.4), must be blowing up at finite time (see [12]), we cannot expect that (1.1)–(1.4) has a global smooth solution with general initial data. The