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Existence and Uniqueness of Positive Radial Solutions for a Class of Quasilinear Elliptic Systems

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Abstract. This article is concerned with the existence and uniqueness of positive radial solutions for a class of quasilinear elliptic system. With some reasonable assumptions on the nonlinear source functions and their coefficients, the existence and the upper and lower bounds of the positive solutions will be provided by using the fixed point theorem and the maximum principle for the quasilinear elliptic system.

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1 Introduction

In this paper, we study the existence and uniqueness of positive solutions for the following quasilinear elliptic system

$$\begin{cases} \triangle_p u + a(|x|)f(v) = 0, & x \in \Omega, \\ \triangle_q v + b(|x|)g(w) = 0, & x \in \Omega, \\ \triangle_m w + c(|x|)h(u) = 0, & x \in \Omega, \\ u = v = w = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the open unit ball in $\mathbf{R}^N$ with $N \geq 1$, p, q, $m > 1$, a, b, c: $[0,\infty) \to (0,\infty)$ are continuous functions and f, g, h: $[0,\infty) \to [0,\infty)$ are continuous and nondecreasing.

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In recent years, much attention has been paid to the existence and uniqueness of solutions for the quasilinear elliptic systems with two equations, in particular, for the problem

$$\begin{cases} \triangle_p u + a(|x|)f(v) = 0, & x \in \Omega, \\ \triangle_q v + b(|x|)g(u) = 0, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

See, for example, ([1–10]) and the reference therein.

When $f(v) = |v|^{\delta-1}v$, $g(u) = |u|^{\mu-1}u$, Guo [4] has proved that the problem (1.2) has at least one positive radial solutions.

More recently, Cui, Yang and Zhang [6] studied (1.2) when $a(x) \equiv \lambda$, $b(x) \equiv \mu$, f, g are smooth functions that are negative at the origin and $f(x) \backsim x^m$, $g(x) \sim x^n$ for x large with $m,n \geq 0$, $mn < (p-1)(q-1)$. By using the fixed point theorem in a cone, the authors obtained the existence and uniqueness of positive solutions for (1.2).

For systems with three equations, Yang [2] studied the following problem

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = a(|x|)u^{\alpha_1}v^{\beta_1}w^{\gamma_1}, & x \in B_R, \\ -\mathrm{div}(|\nabla v|^{q-2}\nabla v) = b(|x|)u^{\alpha_2}v^{\beta_2}w^{\gamma_2}, & x \in B_R, \\ -\mathrm{div}(|\nabla w|^{m-2}\nabla w) = c(|x|)u^{\alpha_3}v^{\beta_3}w^{\gamma_3}, & x \in B_R, \\ u = v = w = 0, & x \in \partial B_R. \end{cases} \tag{1.3}$$

By the blowing up argument and degree theory, the author has proved an existence result of positive solutions and obtained a priori bounds for the positive radial solutions of (1.3).

Compared to the case of systems with two equations, there are some extra difficulties in the study of systems with three or more equations. For example, some systems with two equations could have a variational structure, but not for most systems with three or more equations. The readers can find this difficulty in our result of uniqueness.

Motivated by the above results, we aim to investigate the existence and uniqueness of positive solutions for (1.1) by using the fixed point theorem and the maximum principle for the quasilinear elliptic system. And the readers can find the related results for $p = q = m = 2$ in [11].

Throughout this paper, we suppose a, b, c, f, g, h satisfy the following conditions:

(H_1) Each of the functions f, g, and h (denoted by ψ) satisfies $\psi : [0,\infty) \to [0,\infty)$ is continuous, nondecreasing, C^1 on $(0,\infty)$ and $\lim\limits_{x\to 0^+} \sup x\psi'(x) < \infty$.

(H_2) There exist nonnegative numbers α, β, γ, A, B, C with $A,B,C > 0$, $\alpha\beta\gamma < 1$ such that

$$\lim_{x\to 0^+} \inf \frac{f^{\frac{1}{p-1}}(x)}{x^\alpha} > 0, \quad \lim_{x\to 0^+} \inf \frac{g^{\frac{1}{q-1}}(x)}{x^\beta} > 0, \quad \lim_{x\to 0^+} \inf \frac{h^{\frac{1}{m-1}}(x)}{x^\gamma} > 0,$$