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Inequalities and Separation for a Biharmonic Laplace-Beltrami Differential Operator in a Hilbert Space Associated with the Existence and Uniqueness Theorem

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Abstract. In this paper, we have studied the separation for the biharmonic Laplace-Beltrami differential operator

$$Au(x) = -\Delta\Delta u(x) + V(x)u(x),$$

for all $x \in R^n$, in the Hilbert space $H = L_2(R^n, H_1)$ with the operator potential $V(x) \in C^1(R^n, L(H_1))$, where $L(H_1)$ is the space of all bounded linear operators on the Hilbert space H_1, while $\Delta\Delta u$ is the biharmonic differential operator and

$$\Delta u = -\sum_{i,j=1}^{n} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left[\sqrt{\det g}\, g^{-1}(x) \frac{\partial u}{\partial x_j} \right]$$

is the Laplace-Beltrami differential operator in R^n. Here $g(x) = (g_{ij}(x))$ is the Riemannian matrix, while $g^{-1}(x)$ is the inverse of the matrix $g(x)$. Moreover, we have studied the existence and uniqueness Theorem for the solution of the non-homogeneous biharmonic Laplace-Beltrami differential equation $Au = -\Delta\Delta u + V(x)u(x) = f(x)$ in the Hilbert space H where $f(x) \in H$ as an application of the separation approach.

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1 Introduction

The concept of separation for differential operators was first introduced by Everitt and Giertz [1,2]. They have obtained the separation results for the Sturm Liouville differential

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operator

$$Au(x)=-u''(x)+V(x)u(x), \qquad x\in R, \tag{1.1}$$

in the space $L_2(R)$. They have studied the following question: What are the conditions on $V(x)$ such that if $u(x)\in L_2(R)$ and $Au(x)\in L_2(R)$ imply that both of $u''(x)$ and $V(x)u(x)\in L_2(R)$? More fundamental results of separation for differential operators were obtained by Everitt and Giertz [3,4]. A number of results concerning the property referred to the separation of differential operators was discussed by Biomatov [5], Otelbaev [6], Zettle [7] and Mohamed et al. [8–13]. The separation for the differential operators with the matrix potential was first studied by Bergbaev [14]. Brown [15] has shown that certain properties of positive solutions of disconjugate second order differential expressions imply the separation. Some separation criteria and inequalities associated with linear second order differential operators have been studied by Brown et al. [16,17]. Mohamed et al. [11] have studied the separation property of the Sturm Liouville differential operator

$$Au(x)=-(\mu(x)u')'+V(x)u(x), \qquad x\in R, \tag{1.2}$$

in the Hilbert space $H_p(R)$, $(p=1,2)$, where $V(x)\in L(l_p)$ is an operator potential which is a bounded linear operator on l_p and $\mu(x)\in C^1(R)$ is a positive continuous function on R.

Mohamed et al. [9] have studied the separation property for the linear differential operator

$$Au(x)=(-1)^m D^{2m}u(x)+V(x)u(x), \qquad x\in R, \tag{1.3}$$

in the Banach space $L_p(R)^l$ where $V(x)$ is an $l\times l$ positive hermitian matrix and $D^{2m}=d^{2m}/dx^{2m}$ is the classical differentiation of order $2m$.

Mohamed et al. [12] have studied the separation of the Schrodinger operator

$$Au(x)=-\Delta u(x)+V(x)u(x), \qquad x\in R^n\ , \tag{1.4}$$

with the operator potential $V(x)\in C^1(R^n,L(H_1))$ in the Hilbert space $L_2(R^n,H_1)$ and $\Delta=\sum_{i=1}^n(\partial^2/\partial x_i^2)$ is the Laplace operator in R^n.

Mohamed et al. [13] have studied the separation for the general second order differential operator

$$Au(x)=-\sum_{i,j=1}^{n}\alpha_{ij}(x)D_i^ju(x)+V(x)u(x), \qquad x\in R^n\ , \tag{1.5}$$

in the weighted Hilbert space $L_{2,k}(R^n,H_1)$ with a positive weight function $k(x)$ and the operator potential $V(x)\in C^1(R^n,L(H_1))$ where $\alpha_{ij}\in C^2(R^n)$ and $D_i^j=\partial^2/\partial x_i\partial x_j$.

Zayed et al. [18] have obtained recent results on the separation of linear and nonlinear Schrodinger-type operators with the operator potentials in Banach spaces. Furthermore,