JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **29**, No. 4, pp. 255-268

doi: 10.4208/jpde.v29.n4.1
December 2016

A Five-Dimensional System of Navier-Stokes Equation for a Two-Dimensional Incompressible Fluid on a Torus

WANG Heyuan[1,*] and LI Kaitai[2]

[1] *College of Sciences, Liaoning University of technology, Jin'zhou 121001, China.*
[2] *College of Sciences, Xi'an Jiaotong University, Xi'an 710049, China.*

Received 25 January 2016; Accepted 27 September 2016

Abstract. A five-mode truncation of Navier-Stokes equation for a two-dimensional incompressible fluid on a torus is studied. Its stationary solutions and stability are presented, the existence of attractor and the global stability of the system are discussed. The whole process, which shows a chaos behavior approached through an involved sequence of bifurcations with the changing of Reynolds number, is simulated numerically. Based on numerical simulation results of bifurcation diagram, Lyapunov exponent spectrum, Poincare section, power spectrum and return map of the system are revealed.

AMS Subject Classifications: 65N30, 47H15

Chinese Library Classifications: O175.29

Key Words: Navier-Stokes equation; strange attractor; Lyapunov function; bifurcation; chaos.

1 Introduction

In recent years many researchers have studied the stability and the bifurcation of truncated model of Navier-Stokes equation ([1–5]). A pioneering work is given by Lorenz [6] concerning the Saltzman equations for convection between plates, represented the first attempt to study the partial differential equations that govern a fluid flow through truncation of a suitable Fourier expansion to a finite number of components ("modes"). Later Franceschini et al. [3-5] made a further expansion in this research field to study the Navier-Stokes equations. More precisely, they consider the following plane incompressible Navier-Stokes equation

*Corresponding author. *Email addresses:* `wangheyuan6400@sina.com` (H. Y. Wang), `ktli@xjtu.edu.cn` (K. T. Li)

$$\frac{\partial u}{\partial t}+(u\cdot\nabla)u=-\nabla p+f+\nu\triangle u, \tag{1.1}$$

$$\nabla\cdot u=0, \tag{1.2}$$

$$\int_{T^2} u\mathrm{d}X=0, \tag{1.3}$$

on the torus $T^2=[0,2\pi]\times[0,2\pi]$, the velocity field u, the pressure p and (periodic) volume force f is expanded in Fourier series on a torus, a five-mode and a seven-mode Lorenz equations have been proposed in Refs. 2, 3, then they have studied three-dimensional Navier-Stokes equation in $T^3=[0,2\pi]^3$ and obtained some nonlinear system, and discussed these complex system in detail in [4, 5].

In this paper, we studied the dynamical behavior and the numerical simulation of the five-mode equations obtained in [2]. Dynamical behaviors of this chaotic system, including some basic dynamical properties, bifurcations, and routes to chaos, etc, have been investigated both theoretically and numerically by changing Reynolds number. the existence of attractor and the global stability of the equations have been firmly verified and these theories can be used in other similar model. Furthermore, some dynamical behavior features of the chaos system in a certain range of the Reynolds number, such as bifurcation diagram, Lyapunov exponent spectrum, Poincare section, power spectrum and return map, are presented.

2 The five-mode Lorenz-like equations

We expanded u, f, p in Frourier series :

$$u(X,t)=\sum_{K\neq 0} e^{iK\cdot X} r_K \frac{K^{\perp}}{|K|}, \tag{2.1}$$

$$f(X,t)=\sum_{K\neq 0} e^{iK\cdot X} f_K \frac{K^{\perp}}{|K|}, \tag{2.2}$$

$$p(X,t)=\sum_{K\neq 0} e^{iK\cdot X} p_K \frac{K^{\perp}}{|K|}, \tag{2.3}$$

where $K=(k_1,k_2)$ is a "wave vector", with integer components, $K^{\perp}=(k_2,-k_1)$, $r_K=r_K(t)$ is a function of time t. Substituting (2.1)-(2.3) into (1.1), we get formally the following equations for $\{r_K\}_{K\neq 0}$

$$\dot{r}_K=-i\sum_{K_1+K_2+K=0}\frac{K_1^{\perp}\cdot K_2(K_2^2-K_1^2)}{2|K||K_1||K_2|}\overline{r}_{K_1}\overline{r}_{K_2}-\nu|K|^2 r_K+f_K(K\in L), \tag{2.4}$$

where L is a set of wave vectors, and if $K\in L$, also $-K\in L$.